System of radiative transfer equations for coupled surface and body waves

Maarten V. de Hoop, Josselin Garnier and Knut Sølna

Abstract. We obtain and analyze a system of radiative transfer equations associated with surface and body waves. The system accounts for a boundary along which surface waves propagate and body waves propagating in the bulk. The system describing the wave mode coupling is parameterized by the range coordinate in the direction along the boundary. We distinguish two layers beneath the boundary containing distinct random fluctuations, analyze the coupling of surface and body modes and introduce proper scaling regimes which through diffusion approximation theory leads to the mentioned equations. We present particular properties of the solutions and the qualitative behavior including equipartitioning in the appropriate limit resulting in converting energy from surface to body waves.

Keywords. Radiative transfer; random medium; surface waves.

1. Introduction

Radiative transfer [8] has been used for a long time to model wave propagation in heterogeneous media like Earth's crust [17, 20, 22, 23], biological tissue [2], the atmosphere and the ocean [1, 14]. The mathematical theory of radiative transfer in open random media, which involves only body waves, is well established [3, 4, 6, 5, 13, 19]. However, the coupling between surface waves propagating along boundaries and body waves propagating in the bulk medium remains a challenging problem [15, 18, 21, 24]; this coupling is essential in understanding, for example, seismic coda.

We consider the two-dimensional wave equation either in a half-space $\mathbb{R} \times (0, +\infty)$ or in a finite width section:

$$\frac{n^2(x,z)}{c_o^2}\partial_t^2 u - \Delta u = f(x,z;t), \quad (x,z;t) \in \mathbb{R} \times (0,D) \times \mathbb{R}, \tag{1}$$

with Dirichlet boundary condition at $z = 0$, background wave speed c_o, and index of refraction $n(x,z)$. The index of refraction n is assumed to be randomly heterogeneous with a mean profile that supports guided modes and where the random fluctuations are responsible for wave scattering. We consider in particular the special case when the random fluctuations are supported only in a layer close to the surface $z = 0$. Our goal is to derive from a multiscale analysis the radiative transfer equation (RTE) satisfied by the Wigner transform of the normal derivative of the wave field observed at the surface $z = 0$. This Wigner transform has discrete and continuous components due to the presence of surface- and body-wave modes. A multiscale analysis makes it possible to derive a RTE that expresses the effective coupling between the different wave modes. The inspection of this RTE reveals that in the case of a half-space the effective coupling acts between surface waves and from surface waves to body waves, but there is no effective coupling from the body waves to the surface waves. This was already noticed by Garnier [11] in the context of coupled mode theory and comes from the fact that the latter form of coupling is too weak to be captured by the standard multiscale analysis that has a limited range of validity for the propagation distance.

In this paper, we propose to address the case of a domain of the form of a finite width section $\mathbb{R} \times (0, D)$ and to study the situation in which D becomes large. This approach makes it possible to capture the leading-order terms of the effective coupling between all types of modes. We then get a novel form of the RTE that fully couples surface and body waves and that describes the long-range dynamics towards equipartition.

The energy transport in seismic coda has been studied for decades. Descriptions in terms of radiative transfer date back to Wu [23]. For a comprehensive monograph, we refer to Sato, Fehler and Maeda [20]. Concerning the consideration of coupling surface and body waves, we mention a few key developments that motivated our work.

In the case of a slab bounded by two free surfaces, Trégourès and Van Tiggelen [22] derived from first principles a quasi two-dimensional radiative transfer equation where the wavefield is expanded in a basis of Rayleigh, Love and Lamb eigenmodes. Through this normal-mode decomposition, this model incorporates the boundary conditions exactly. The energy exchange between surface and body waves was treated by normal-mode coupling in the Born approximation. This formulation enabled the prediction of energy decay in the coda and its partitioning into different components. Zeng [24] introduced a system of coupled integral equations to describe the exchange of energy between surface waves and body waves in the seismic coda. However, the underlying arguments are purely phenomenological. Maeda, Sato and Nishimura [15] presented a study of energy exchange between surface and body waves starting from the elastodynamic equations in a half space. Using the Born approximation, they evaluated the scattering coefficients between all modes of propagation in a medium containing random inhomogeneities.

To describe the energy transport in the seismic coda, Margerin, Bajaras and Campillo [18] introduced a system of radiative transfer equations for coupled surface and body waves in a scalar approximation. They identified cross sections for surface-to-body and body-to-surface waves scattering. They followed a phenomenological approach to obtain the specific energy density of surface and body waves in a medium containing a homogeneous distribution of point scatterers.

Our approach is based on a mathematical model involving a thick waveguide with a thin surface layer and random heterogeneities makes it possible to derive a RTE that is valid for long distances and that captures the coupling between surface and body waves. A preliminary and reduced version of this approach was applied in the context of coupled mode theory in Ref. [7]. Here we address a general form of the index of refraction and we give a complete description of the discrete and continuous components of the Wigner transform. This gives a detailed description of the dynamics towards equipartition. Moreover, the energy fluxes in the coda, or specific intensities, are directly predicted by the radiative transfer model. Their angular distribution is of direct importance in imaging applications.

The paper is organized as follows. In Section 2 we address the case of a half space with a thin layer. In Section 3 we address the case of a domain of the form $\mathbb{R} \times (0, D)$ with a mean index of refraction that is constant in the bulk medium and we study the situation in which D becomes large. In Section 4 we consider a general form of the index of refraction.

2. Propagation in a half space containing a thin layer

Considering (1) and standardly taking the Fourier transform,

$$\hat{u}(x, z; \omega) = \int_{\mathbb{R}} u(x, z; t) \exp(i\omega t) dt,$$

we obtain the two-dimensional Helmholtz equation

$$\Delta \hat{u} + k^2 n^2(x, z) \hat{u} = -\hat{f}(x, z; \omega), \quad (x, z) \in \mathbb{R} \times (0, +\infty), \tag{2}$$

with wavenumber $k = \omega / c_o$. We introduce a thin layer through the index of refraction beneath the boundary at $z = 0$ that supports surface modes without and with scattering.

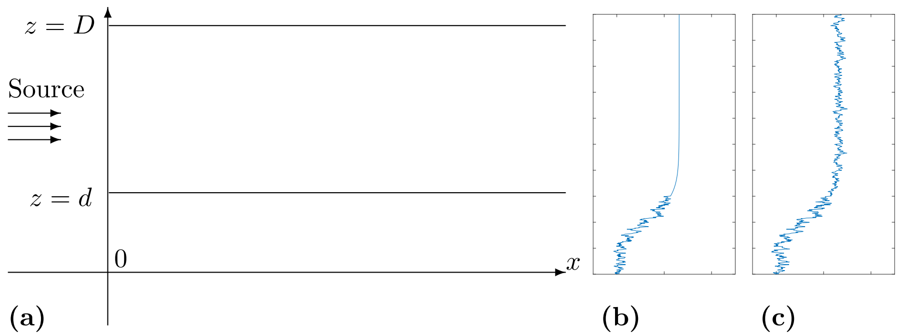

FIGURE 1. The figure illustrates the geometry of the wave propagation scenario considered in the paper. The surface modes are essentially confined to the section $z \in (0, d)$ and the body modes to the section $z \in (0, D)$. We consider here the case with D finite. The propagation is in the x direction and the source is on the left (picture a). We observe the wave field at the surface $z = 0$ and at large range and over a small aperture. The pictures (b-c) show two examples of two randomly perturbed profiles of the index of refraction. The index of refraction has a deterministic background profile that is decreasing in z and is independent of the x coordinate and on top of this profile there are random fluctuations which vary with respect to both x and z coordinates. In the picture (b), resp. (c), the random fluctuations are supported in the surface layer, resp. in the full waveguide cross section.

2.1. Non-scattering thin layer

When the thin layer is non-scattering, the index of refraction is x-independent and equal to $\mathrm{n_b}(z)$. The function $\mathrm{n_b}(z)$ is such that $\mathrm{n_b}(0) = n_0$, $\mathrm{n_b}(z)$ is non-increasing on $[0, d]$ from n_0 to $n_1 < n_0$, and $\mathrm{n_b}(z) = n_1$ for $z \geq d$. The configuration is illustrated in Figure 1. The two right plots in the figure show examples of index of refraction. On top of the smooth background component $n_{\mathrm{b}}(z)$ there are random fluctuations. The two right plots illustrate cases where the random fluctuations are supported in the full interval $0 < z < D$ (figure 1c) or only in the section $0 < z < d$ corresponding to the surface layer with a relatively larger values for the index of refraction (figure 1b). As we will see below, even though the random fluctuations are supported only in the surface layer the random fluctuations give a coupling in between modes so that wave power that initially may be carried only by a set of wave surface modes is transmitted to wave body modes supported in the full interval $0 < z < D$ giving a sort of equipartition over modes.

Let us now fix the wavenumber k. The spectral problem associated to the one-dimensional Schrödinger operator $(\partial_z^2 + k^2 \mathrm{n_b^2}(z))\phi(z) = \gamma\phi(z)$ with Dirichlet boundary condition at $z = 0$ (moreover, here defined relative to the smooth background index of refraction component n_{b}) has been well studied (see Magnanini and Santosa [16] and Appendix A):

- The spectrum is of the form $(-\infty, n_1^2 k^2) \cup \{\beta_{N-1}^2, \ldots, \beta_0^2\}$.
- The N modal wavenumbers β_j are positive and $n_1^2 k^2 < \beta_{N-1}^2 < \cdots < \beta_0^2 < n_0^2 k^2$. We have $N \geq 1$ when ω is large enough.
- The functions ϕ_j, $j = 0, \ldots, N-1$, are the modes corresponding to the discrete spectrum. They decay exponentially in z for $z > d$.
- The functions ϕ_γ, $\gamma \in (-\infty, n_1^2 k^2)$, are the modes corresponding to the continuous spectrum. They are oscillatory and bounded at infinity.

- The set of modes is complete in $L^2(0,+\infty)$. Any function $v \in L^2(0,+\infty)$ can be expanded on this complete set:

$$v(z) = \sum_{j=0}^{N-1} v_j \phi_j(z) + \int_{-\infty}^{n_1^2 k^2} v_\gamma \phi_\gamma(z) d\gamma, \tag{3}$$

with $v_j = (\phi_j, v)_{L^2}$ and $v_\gamma = (\phi_\gamma, v)_{L^2}$. We have an isometry from $L^2(\mathbb{R})$ onto $\mathbb{C}^N \times L^2(-\infty, n_1^2 k^2)$ with

$$(v,v)_{L^2} = \sum_{j=0}^{N-1} \big| (\phi_j, v)_{L^2} \big|^2 + \int_{-\infty}^{n_1^2 k^2} \big| (\phi_\gamma, v)_{L^2} \big|^2 d\gamma. \tag{4}$$

- We note that ϕ_γ does not belong to $L^2(0,+\infty)$, but $(\phi_\gamma, v)_{L^2}$ can be defined for any test function $v \in L^2(0,+\infty)$ as

$$(\phi_\gamma, v)_{L^2} = \lim_{D \to +\infty} \int_0^D \phi_\gamma(z) v(z) dz, \tag{5}$$

where the limit holds (as a function in γ) in $L^2(-\infty, n_1^2 k^2)$.

Any function can be expanded on the complete set of the eigenfunctions of the Schrödinger operator. In particular, the solution of the Helmholtz equation (2) can be expanded as the superposition of modes:

$$\hat{u}(x,z) = \sum_{j=0}^{N-1} \hat{u}_j(x) \phi_j(z) + \int_{-\infty}^{n_1^2 k^2} \hat{u}_\gamma(x) \phi_\gamma(z) d\gamma. \tag{6}$$

The modes for $j = 0, \dots, N-1$ are guided, the modes for $\gamma \in (0, k^2)$ are radiating, the modes for $\gamma \in (-\infty, 0)$ are evanescent. Indeed, the complex mode amplitudes satisfy,

$$\partial_x^2 \hat{u}_j + \beta_j^2 \hat{u}_j = 0, \quad j = 0, \dots, N-1, \tag{7}$$

$$\partial_x^2 \hat{u}_\gamma + \gamma \hat{u}_\gamma = 0, \quad \gamma \in (-\infty, n_1^2 k^2), \tag{8}$$

for any x which is not in the support of $\hat{f}$. Therefore, if the source is concentrated on the line $x = 0$ and of the form $\hat{f}(x,z) = \delta(x) F(z)$, then we have for $x > 0$,

$$\hat{u}(x,z) = \sum_{j=0}^{N-1} \frac{a_{j,\mathrm{s}}}{\sqrt{\beta_j}} e^{i\beta_j x} \phi_j(z) + \int_0^{n_1^2 k^2} \frac{a_{\gamma,\mathrm{s}}}{\gamma^{1/4}} e^{i\sqrt{\gamma} x} \phi_\gamma(z) d\gamma + \int_{-\infty}^0 \frac{a_{\gamma,\mathrm{s}}}{|\gamma|^{1/4}} e^{-\sqrt{|\gamma|} x} \phi_\gamma(z) d\gamma, \tag{9}$$

where the mode amplitudes are constant and determined by the source,

$$a_{j,\mathrm{s}} = \frac{\sqrt{\beta_j}}{2} \int_0^\infty \phi_j(z) F(z) dz, \quad j = 0, \dots, N-1, \tag{10}$$

$$a_{\gamma,\mathrm{s}} = \frac{|\gamma|^{1/4}}{2} \int_0^\infty \phi_\gamma(z) F(z) dz, \quad \gamma \in (-\infty, n_1^2 k^2). \tag{11}$$

For x much larger than the wavelength and $z \in (0, d)$, the leading-order terms are the guided mode amplitudes:

$$\hat{u}(x,z) = \sum_{j=0}^{N-1} \frac{a_{j,\mathrm{s}}}{\sqrt{\beta_j}} e^{i\beta_j x} \phi_j(z) + O\left(\frac{1}{kx}\right). \tag{12}$$

2.2. Scattering thin layer

When the thin layer is scattering, u satisfies (1) with a randomly perturbed index of refraction:

$$n^2(x,z) = \mathrm{n}_\mathrm{b}^2(z) + \varepsilon \nu(x,z), \quad z \in (0,+\infty), \tag{13}$$

where

- the function $\mathrm{n}_\mathrm{b}(z)$ is such that $\mathrm{n}_\mathrm{b}(0) = n_0$, $\mathrm{n}_\mathrm{b}(z)$ is nonincreasing on $[0,d]$ from n_0 to $n_1 < n_0$, and $\mathrm{n}_\mathrm{b}(z) = n_1$ for $z \geq d$;

- ν is a zero-mean, bounded, random process, stationary in x, compactly supported in z, so that $\mathbb{E}[\nu(x+x',z)\nu(x',z')] = 0$ as soon as $\max(z,z') > d'$ for some $d' > 0$. As a function of x, the covariance function $\mathbb{E}[\nu(x+x',z)\nu(x',z')] = \mathbb{E}[\nu(x,z)\nu(0,z')]$ is assumed to decay fast enough at infinity so that a diffusion-approximation theorem [10, Chapter 10] can be used and it should be smooth enough, so that the forward-scattering approximation that is addressed in Ref. [12] is satisfied. A typical example is a Gaussian covariance function $\mathbb{E}[\nu(x+x'z)\nu(x',z')] = \mathcal{R}(z,z')\exp(-x^2/\ell^2)$.

The asymptotic analysis of the wave field and its moments follows an earlier analysis of one the authors (Ref. [11]). It is possible to write a radiative transfer equation for the incoherent wave fluctuations. The Wigner transform of the normal derivative of the field at the surface is defined as a distribution as a local Fourier transform of the covariance function of the normal derivative of the field by

$$\begin{aligned} W^{\mathrm{s}}(x,\kappa;t,\omega) &= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\iint d\omega' dx' \exp(-i\omega' t - i\kappa x') \\ &\quad\times \mathbb{E}\Big[\partial_z \hat{u}(\frac{x}{\varepsilon^2}+\frac{x'}{2},0;\omega+\frac{\varepsilon^2}{2}\omega')\partial_z\overline{\hat{u}}(\frac{x}{\varepsilon^2}-\frac{x'}{2},0;\omega-\frac{\varepsilon^2}{2}\omega')\Big] \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\iint dt' dx' \exp(i\omega t' - i\kappa x') \\ &\quad\times \mathbb{E}\Big[\partial_z u(\frac{x}{\varepsilon^2}+\frac{x'}{2},0;\frac{t}{\varepsilon^2}+\frac{t'}{2})\partial_z u(\frac{x}{\varepsilon^2}-\frac{x'}{2},0;\frac{t}{\varepsilon^2}-\frac{t'}{2})\Big]. \end{aligned} \tag{14}$$

It represents the energy density at time t and frequency ω that arrives at x with the angle determined by the longitudinal wavenumber κ. Then

Proposition 2.1. *The Wigner transform W^{s} of the incoherent wave is of the form:*

$$W^{\mathrm{s}}(x,\kappa;t,\omega) = \sum_{j=0}^{N(\omega)-1} W_j^{\mathrm{s}}(x;t,\omega)\delta(\kappa-\beta_j(\omega)), \tag{15}$$

for $\kappa \in (n_1 k(\omega), n_0 k(\omega))$, $k(\omega) = \omega/c_o$. The $W_j^{\mathrm{s}}(x;t,\omega)$'s satisfy

$$\partial_x W_j^{\mathrm{s}} + \frac{1}{v_j(\omega)}\partial_t W_j^{\mathrm{s}} = \sum_{l=0,l\neq j}^{N(\omega)-1} \Gamma_{jl}^{\mathrm{s}}(\omega) W_l^{\mathrm{s}} - \Lambda_j^{\mathrm{s}}(\omega) W_j^{\mathrm{s}}, \tag{16}$$

where

$$v_j(\omega) = \frac{1}{\beta_j'(\omega)}, \tag{17}$$

$$\Gamma_{jl}^{\mathrm{s}}(\omega) = \frac{\beta_l(\omega)\partial_z\phi_j(0;\omega)^2}{\beta_j(\omega)\partial_z\phi_l(0;\omega)^2}\Gamma_{jl}^{c}(\omega), \tag{18}$$

$$\Gamma_{jl}^{c}(\omega) = \frac{k^4(\omega)}{2\beta_j\beta_l(\omega)}\int_0^\infty \mathcal{R}_{jl}(x;\omega)\cos\big((\beta_l(\omega)-\beta_j(\omega))x\big)dx, \tag{19}$$

$$\mathcal{R}_{jl}(x;\omega) = \int_0^\infty\int_0^\infty \phi_j\phi_l(z;\omega)\mathbb{E}[\nu(0,z)\nu(x,z')]\phi_j\phi_l(z';\omega)dzdz', \tag{20}$$

$$\begin{aligned} \Lambda_j^{\mathrm{s}}(\omega) &= \sum_{l=0,l\neq j}^{N(\omega)-1}\Gamma_{jl}^{c}(\omega) \\ &\quad + \int_0^{n_1^2 k^2(\omega)} \frac{k^4(\omega)}{2\sqrt{\gamma}\beta_j(\omega)}\int_0^\infty \mathcal{R}_{j\gamma}(x;\omega)\cos\big((\sqrt{\gamma}-\beta_j(\omega))x\big)dxd\gamma, \end{aligned} \tag{21}$$

$$\mathcal{R}_{j\gamma}(x;\omega) = \int_0^\infty\int_0^\infty \phi_j\phi_\gamma(z;\omega)\mathbb{E}[\nu(0,z)\nu(x,z')]\phi_j\phi_\gamma(z';\omega)dzdz'. \tag{22}$$

This proposition (proved in Appendix B.1) shows conversion between surface modes and irreversible leakage of surface wave power towards body waves. The complete form of the Wigner transform is actually more complicated than the one described in the proposition, as it also contains coherent contributions that decay exponentially with the propagation distance. The full expression of the Wigner transform is given by (105) in Appendix B.1. In the RTE, v_j is the group velocity of the j-th surface mode, $\Gamma^{\rm s}_{jl}$ is the scattering cross coefficient (energy coming from the l-th surface mode to the j-th surface mode), and $\Lambda^{\rm s}_j$ is the extinction coefficient, that takes into account leakage towards the body modes and scattering to other surface modes. The scattering and extinction coefficients depend on the two-point statistics of the fluctuations of the random medium.

3. Propagation in a waveguide with a thin layer

When we proceed from (2) with (13), then the effective radiative transfer system does not include conversion from the radiating (body) modes to the guided (surface) modes, while only conversion between surface modes and leakage from surface modes to body modes is manifest (Ref. [11]). Here, we aim to generate a RTE that contains coupling between all the different types of modes. To do so, we need to modify (1) with (13) in an essential way. For the analysis we use an approach that consists of considering a truncated problem with a domain of the form $\mathbb{R}\times(0,D)$ with a large $D>0$. Such an approach with a sequence of truncated problems has been used in the homogeneous case to prove the completeness of the set of modes in the half-space system, and to relate the eigenvalues and eigenfunctions of the truncated problems to the ones of the half-space problem (employing the Levitan-Levinson transform method, see Chapter 10 in Coddington and Levinson [9] or Appendix A). It was also recently applied in Ref. [11] to study the evolution of the mean mode powers of the incoherent wave field.

3.1. Scattering waveguide

Let $D>d$. We consider the truncated problem

$$\Delta\hat{u}+k^2n^2(x,z)\hat{u}=-\hat{f}(x,z;\omega),\quad (x,z)\in\mathbb{R}\times(0,D), \tag{23}$$

with $k=\omega/c_o$, Dirichlet boundary condition at $z=0$ and Neumann boundary condition at $z=D$, and

$$n^2(x,z)={\rm n}_{\rm b}^2(z)+\varepsilon\nu(x,z)\text{ if } z\in(0,D), \tag{24}$$

where

- the function ${\rm n}_{\rm b}(z)$ is such that ${\rm n}_{\rm b}(0)=n_0$, ${\rm n}_{\rm b}(z)$ is non-increasing on $[0,d]$ from n_0 to $n_1<n_0$, and ${\rm n}_{\rm b}(z)=n_1$ for $z\geq d$ (see figure 1);
- ν is a zero-mean random process, stationary in x, compactly supported in z, so that $\mathbb{E}[\nu(x,z)\nu(x',z')]=0$ as soon as $\max(z,z')>d'$ for some $d'>0$.

For any fixed D, the spectral problem associated to the one-dimensional Schrödinger operator $(\partial_z^2+k^2{\rm n}_{\rm b}^2(z))\phi(z)=\gamma\phi(z)$ in $L^2(0,D)$ with Dirichet boundary condition at $z=0$ and Neumann boundary condition at $z=D$ is fully understood. The spectrum is discrete. The eigenvalues are of the form $\gamma_{j,D}$ with $\cdots<\gamma_{j+1,D}<\gamma_{j,D}<\cdots<\gamma_{0,D}<n_0^2k^2$. We denote N_D such that $\gamma_{N_D,D}\leq n_1^2k^2<\gamma_{N_D-1,D}$ and M_D such that $\gamma_{M_D,D}\leq 0<\gamma_{M_D-1,D}$. For $j\leq M_D$ we write $\beta_{j,D}(\omega)=\sqrt{\gamma_{j,D}}$. The eigenfunctions are functions $\phi_{j,D}\in L^2(0,D)$. These functions are exponentially decaying in (d,D) for $j<N_D$ and oscillatory for $j\geq N_D$. The set of eigenfunctions is complete in $L^2(0,D)$.

We can write a radiative transfer type equation for the truncated problem with a fixed D as $\varepsilon\to 0$. In this truncated problem there are only discrete modes and the Wigner transform is discrete. We have

Proposition 3.1. *In the regime $\varepsilon \to 0$, the Wigner transform (14) has the form*

$$W_D^{\rm s}(x,\kappa;t,\omega) = \sum_{j=0}^{M_D(\omega)-1} W_{j,D}^{\rm s}(x;t,\omega)\delta\big(\kappa - \beta_{j,D}(\omega)\big) \tag{25}$$

for $\kappa \in (0, n_0 k(\omega))$, where the $W_{j,D}^{\rm s}(x;t,\omega)$'s are of the form

$$W_{j,D}^{\rm s}(x;t,\omega) = \frac{\partial_z \phi_{j,D}(0;\omega)^2}{\beta_{j,D}(\omega)} W_{j,D}(x;t,\omega) \tag{26}$$

and the $W_{j,D}(x;t,\omega)$'s satisfy

$$\partial_x W_{j,D} + \frac{1}{v_{j,D}(\omega)}\partial_t W_{j,D} = \sum_{l=0,l\neq j}^{M_D(\omega)-1} \Gamma_{jl,D}^c(\omega) W_{l,D} - \Lambda_{j,D}(\omega) W_{j,D}, \tag{27}$$

where

$$v_{j,D}(\omega) = \frac{1}{\beta_{j,D}'(\omega)}, \tag{28}$$

$$\Gamma_{jl,D}^c(\omega) = \frac{k^4(\omega)}{2\beta_{j,D}\beta_{l,D}(\omega)} \int_0^\infty \mathcal{R}_{jl,D}(x;\omega) \cos\big((\beta_{l,D}(\omega) - \beta_{j,D}(\omega))x\big) dx, \quad j \neq l, \tag{29}$$

$$\mathcal{R}_{jl,D}(x;\omega) = \int_0^D \int_0^D \phi_{j,D}\phi_{l,D}(z;\omega) \mathbb{E}[\nu(0,z)\nu(x,z')] \phi_{j,D}\phi_{l,D}(z';\omega) dz dz',$$

$$\Lambda_{j,D}(\omega) = \sum_{l=0,l\neq j}^{M_D(\omega)-1} \Gamma_{jl,D}^c(\omega). \tag{30}$$

This is a standard RTE for a closed waveguide [10, Chapter 20].

3.2. A thick scattering waveguide containing a thin layer

We next analyze a configuration of the type considered in the previous subsection when kD becomes large. We now assume that the index of refraction is of the form

$$n^2(x,z) = {\rm n}_{\rm b}^2(z) + \varepsilon \nu(x,z) + \frac{\varepsilon\sqrt{D_0}}{\sqrt{D}} \mu(x,z) \text{ if } z \in (0,D), \tag{31}$$

where

- the function ${\rm n}_{\rm b}(z)$ is such that ${\rm n}_{\rm b}(0) = n_0$, ${\rm n}_{\rm b}(z)$ is non-increasing on $[0,d]$ from n_0 to $n_1 < n_0$, and ${\rm n}_{\rm b}(z) = n_1$ for $z \geq d$;
- ν is a zero-mean, bounded, random process, stationary in x, compactly supported in z, so that $\mathbb{E}[\nu(x,z)\nu(x',z')] = 0$ as soon as $\max(z,z') > d'$ for some d', and satisfying the hypotheses set forth in the previous section;
- μ is a zero-mean, bounded, random process, stationary in x and z, independent from ν. D_0 is a reference length that is added so that μ is dimensionless. We may have $\mu = 0$. We denote

  $$\mathbb{E}[\mu(x,z)\mu(x',z')] = \mathcal{R}^\mu(x-x', z-z').$$

  The covariance function $\mathcal{R}^\mu$ of the process μ should satisfy the same correlation and smoothness properties as the one of ν.

When $kD \gg 1$ (see Appendix A), the situation becomes similar to the half-space problem addressed in Subsections 2.1-2.2: The discrete eigenvalues $\beta_{j,D}^2(\omega)$ and eigenfunctions $\phi_{j,D}(z;\omega)$ for $j < N$ converge when $kD \to +\infty$ to the discrete eigenvalues $\beta_j^2(\omega)$ and eigenfunctions $\phi_j(z;\omega)$ of the half-space problem discussed in Subsection 2.1.

The eigenvalues $\beta_{j,D}^2(\omega)$ for $N \leq j < M_D$ become more numerous and denser as kD increases. We have $\beta_{j,D} \simeq k\mathcal{B}\big((j-N)/(kD)\big)$, where $\mathcal{B}(b)$ is the dimensionless function defined by:

$$\mathcal{B}(b) := \sqrt{n_1^2 - \pi^2 b^2}\,\mathbf{1}_{(0,n_1)}(\pi b),$$

for $j = N, \dots, M_D$ with $M_D - N \simeq n_1 kD/\pi$. The eigenfunctions have the form

$$\phi_{j,D}(z;\omega) \simeq \frac{\sqrt{k}}{\sqrt{D\mathcal{N}(\gamma/k^2)}}\phi_\gamma(z;\omega),$$

where $\gamma = \beta_{j,D}^2$, ϕ_γ is a mode corresponding to the continuous spectrum of the half-space problem discussed in Subsection 2.1, and $\mathcal{N}(g)$ is the normalized density of states (function of a normalized squared longitudinal wavenumber $g = \gamma/k^2$):

$$\mathcal{N}(g) := \frac{1}{2\pi\sqrt{n_1^2 - g}}\mathbf{1}_{(0,n_1^2)}(g).$$

This means that the number of eigenvalues $\beta_{j,D}^2$ in $(\gamma, \gamma+\delta\gamma)$ is $(D/k)\mathcal{N}(\gamma/k^2)\delta\gamma$ for small $\delta\gamma$. We now present the main result of this paper in

Proposition 3.2. *When $kD \gg 1$, the Wigner transform (14) of the normal derivative of the field at the surface is of the form*

$$W_D^{\rm s}(x,\kappa;t,\omega) \simeq \sum_{j=0}^{N-1} \frac{\partial_z\phi_j(0;\omega)^2}{\beta_j(\omega)} W_j(x;t,\omega)\delta\big(\kappa - \beta_j(\omega)\big) + \frac{\partial_z\tilde{\phi}_\xi(0;\omega)^2}{k\xi}\tilde{\mathcal{N}}(\xi)\tilde{W}_\xi(x;t,\omega)\big|_{\xi=\kappa/k}, \tag{32}$$

and the Wigner transforms $(W_j, \tilde{W}_\xi)$ satisfy the coupled radiative transfer equations

$$\partial_x W_j + \frac{1}{v_j}\partial_t W_j = \sum_{l=0, l\neq j}^{N-1} \Gamma_{jl}^c W_l + \int_0^\infty \tilde{\Gamma}_{j\xi'}^c \tilde{W}_{\xi'}\tilde{\mathcal{N}}(\xi')d\xi' - \Lambda_j^c W_j, \tag{33}$$

$$\partial_x \tilde{W}_\xi + \frac{1}{v_\xi}\partial_t \tilde{W}_\xi = \frac{1}{kD}\sum_{l=0}^{N-1} \tilde{\Gamma}_{\xi l}^c W_l + \frac{1}{kD}\int_0^\infty \tilde{\Gamma}_{\xi\xi'}^c \tilde{W}_{\xi'}\tilde{\mathcal{N}}(\xi')d\xi' - \frac{1}{kD}\tilde{\Lambda}_\xi^c \tilde{W}_\xi, \tag{34}$$

for $j = 0, \dots, N-1$ and $\xi \in (0, n_1)$, where the group velocities are

$$v_j(\omega) = \frac{1}{\beta_j'(\omega)}, \qquad v_\xi = \frac{c_o\xi}{n_1^2}, \tag{35}$$

the normalized density of states is

$$\tilde{\mathcal{N}}(\xi) = 2\xi\mathcal{N}(\xi^2) = \frac{\xi}{\pi\sqrt{n_1^2 - \xi^2}}\mathbf{1}_{(0,n_1)}(\xi), \tag{36}$$

the differential scattering cross sections are

$$\Gamma_{jl}^c(\omega) = \frac{k^4}{2\beta_j\beta_l(\omega)}\int_0^\infty \mathcal{R}_{jl}(x;\omega)\cos\big((\beta_l(\omega) - \beta_j(\omega))x\big)dx, \tag{37}$$

$$\tilde{\Gamma}_{j\xi}^c(\omega) = \frac{k^3}{2\beta_j(\omega)\xi}\int_0^\infty \tilde{\mathcal{R}}_{j\xi}(x;\omega)\cos\big((k\xi - \beta_j(\omega))x\big)dx, \tag{38}$$

$$\tilde{\Gamma}_{\xi\xi'}^c(\omega) = \frac{k^2}{2\xi\xi'}\int_0^\infty \big[\tilde{\mathcal{R}}_{\xi\xi'}(x;\omega) + \tilde{\mathcal{R}}_{\xi\xi'}^\mu(x;\omega)\big]\cos\big(k(\xi - \xi')x\big)dx, \tag{39}$$

the total scattering cross sections are

$$\Lambda_j^c(\omega) = \sum_{l=0,l\neq j}^{N(\omega)-1} \Gamma_{jl}^c(\omega) + \int_0^\infty \tilde{\Gamma}_{j\xi'}^c(\omega)\tilde{\mathcal{N}}(\xi')d\xi', \tag{40}$$

$$\tilde{\Lambda}_\xi^c(\omega) = \sum_{l=0}^{N(\omega)-1} \tilde{\Gamma}_{\xi l}^c(\omega) + \int_0^\infty \tilde{\Gamma}_{\xi\xi'}^c(\omega)\tilde{\mathcal{N}}(\xi')d\xi', \tag{41}$$

the correlation functions are defined by

$$\mathcal{R}_{jl}(x;\omega;\omega) = \int_0^\infty \int_0^\infty \phi_j\phi_l(z;\omega)\mathbb{E}[\nu(0,z)\nu(x,z')]\phi_j\phi_l(z';\omega)dzdz', \tag{42}$$

$$\tilde{\mathcal{R}}_{j\xi}(x) = \int_0^\infty \int_0^\infty \phi_j\tilde{\phi}_\xi(z;\omega)\mathbb{E}[\nu(0,z)\nu(x,z')]\phi_j\tilde{\phi}_\xi(z';\omega)dzdz', \tag{43}$$

$$\tilde{\mathcal{R}}_{\xi\xi'}(x;\omega) = \int_0^\infty \int_0^\infty \tilde{\phi}_\xi\tilde{\phi}_{\xi'}(z;\omega)\mathbb{E}[\nu(0,z)\nu(x,z')]\tilde{\phi}_\xi\tilde{\phi}_{\xi'}(z';\omega)dzdz', \tag{44}$$

$$\tilde{\mathcal{R}}_{\xi\xi'}^\mu(x;\omega) = k^2 D_0 \int_0^\infty \mathcal{R}^\mu(x,z)\cos\left(kz\sqrt{n_1^2-\xi^2}\right)\cos\left(kz\sqrt{n_1^2-\xi'^2}\right)dz, \tag{45}$$

where for $\xi \in (0, n_1)$,

$$\tilde{\phi}_\xi(z;\omega) = \frac{1}{\sqrt{\mathcal{N}(\xi^2)}}\phi_{k^2\xi^2}(z;\omega), \tag{46}$$

with ϕ_γ, $\gamma = k^2\xi^2$, *a mode corresponding to the continuous spectrum of the half-space problem defined by (91).*

Proof. The proof consists in carrying out an asymptotic analysis as $D \to +\infty$ of the statement of Proposition 3.1. The technical details are presented in Appendix B.2. □

3.3. The properties of solutions of the coupled radiative transfer equations (33)-(34)

In this subsection, we analyze the properties of the Wigner transform as described by Proposition 3.2.

First, we note that the mean mode powers,

$$P_j(x;\omega) = \int_{-\infty}^\infty W_j(x;\omega,t)dt, \quad \tilde{P}_\xi(x;\omega) = \int_{-\infty}^\infty \tilde{W}_\xi(x;\omega,t)dt, \tag{47}$$

satisfy

$$\partial_x P_j = \sum_{l=0,l\neq j}^{N-1} \Gamma_{jl}^c P_l + \int_0^\infty \tilde{\Gamma}_{j\xi'}^c \tilde{P}_{\xi'}\tilde{\mathcal{N}}(\xi')d\xi' - \Lambda_j^c P_j, \tag{48}$$

$$\partial_x \tilde{P}_\xi = \frac{1}{kD}\sum_{l=0}^{N-1} \tilde{\Gamma}_{\xi l}^c P_l + \frac{1}{kD}\int_0^\infty \tilde{\Gamma}_{\xi\xi'}^c \tilde{P}_{\xi'}\tilde{\mathcal{N}}(\xi')d\xi' - \frac{1}{kD}\tilde{\Lambda}_\xi^c \tilde{P}_\xi. \tag{49}$$

Second, the total power,

$$\mathcal{P}(x;\omega) = \sum_{j=0}^{N(\omega)-1} P_j(x;\omega) + kD\int_0^\infty \tilde{P}_\xi(x;\omega)\tilde{\mathcal{N}}(\xi)d\xi, \tag{50}$$

is a conserved quantity, that is, $\partial_x \mathcal{P} = 0$.

Third, the parameters Γ^c and $\tilde{\Gamma}^c$ are of order $k^2\sigma^2\ell_c$ where σ and ℓ_c are the standard deviation and the correlation length of the random fluctuations, respectively. If the source only generates surface waves, then the radiative transfer equation can be reduced to

$$\partial_x W_j + \frac{1}{v_j}\partial_t W_j = \sum_{l=0,l\neq j}^{N-1} \Gamma_{jl}^c W_l - \Lambda_j^c W_j$$

for propagation distances of the order of $1/(k^2\sigma^2\ell_c)$, which is the equation determined in the half-space case discussed in Subsection 2.2. Indeed we may check that

$$\Lambda_j^c(\omega) = \sum_{l=0,l\neq j}^{N-1} \Gamma_{jl}^c(\omega) + \int_0^{n_1^2k^2} \frac{k^4}{2\sqrt{\gamma}\beta_j(\omega)} \int_0^\infty \mathcal{R}_{j\gamma}(x;\omega)\cos\big((\sqrt{\gamma}-\beta_j(\omega))x\big)dxd\gamma.$$

The power is initially carried by the surface modes. Coupling induces changes in the distribution amongst the surface modes and a decay, which gives for x of the order of $1/(k^2\sigma^2\ell_c)$,

$$\big(P_j(x)\big)_{j=0}^{N-1} \simeq \exp(-\mathbf{M}x)\big(P_j(0)\big)_{j=0}^{N-1}, \tag{51}$$

where $\mathbf{M}$ is the positive matrix with entries

$$M_{jl} = \Lambda_j^c\delta_{jl} - \Gamma_{jl}^c. \tag{52}$$

The decay is in fact a transfer of power from the surface modes to the body modes, expressed by

$$\partial_x\tilde{P}_\xi = \frac{1}{kD}\sum_{l=0}^{N-1}\tilde{\Gamma}_{\xi l}^c P_l, \quad \tilde{P}_\xi(0) = 0,$$

which gives for x of the order of $1/(k^2\sigma^2\ell_c)$,

$$\tilde{P}_\xi(x) = \frac{1}{kD}\sum_{l,l'=0}^{N-1}\tilde{\Gamma}_{\xi l}^c\big(\mathbf{M}^{-1}(\mathbf{I}-\exp(-\mathbf{M}x)\big)_{ll'}P_{l'}(0). \tag{53}$$

Fourth, for propagation distances x of the order of $D/(k\sigma^2\ell_c)$ the mean powers of the surface modes and body modes become of the order of $\mathcal{P}(0)/(kD)$ with $\mathcal{P}(0) = \sum_{j=0}^{N-1}P_j(0)$, and the full equations (48)-(49) should be considered. These equations show that the surface mode powers are in a quasi-equilibrium state that is determined by the body mode power distribution. We have

$$(P_j(x))_{j=0}^{N-1} = \mathbf{M}^{-1}\left(\int_0^\infty \tilde{\Gamma}_{j\xi'}^c\tilde{P}_{\xi'}(x)\tilde{\mathcal{N}}(\xi')d\xi'\right)_{j=0}^{N-1}. \tag{54}$$

The mean body mode powers $\tilde{P}_\xi$ slowly evolve at the scale $D/(k\sigma^2\ell_c)$ and satisfy the equation

$$\partial_x\tilde{P}_\xi = \frac{1}{kD}\sum_{l,l'=0}^{N-1}\int_0^\infty\tilde{\Gamma}_{\xi l}^c(\mathbf{M}^{-1})_{ll'}\tilde{\Gamma}_{l'\xi'}^c\tilde{P}_{\xi'}\tilde{\mathcal{N}}(\xi')d\xi' + \frac{1}{kD}\int_0^\infty\tilde{\Gamma}_{\xi\xi'}^c\tilde{P}_{\xi'}\tilde{\mathcal{N}}(\xi')d\xi' - \frac{1}{kD}\tilde{\Lambda}_\xi^c\tilde{P}_\xi,$$

starting from (for $1/(k^2\sigma^2\ell_c) \ll x_0 \ll D/(k\sigma^2\ell_c)$)

$$\tilde{P}_\xi(x_0) = \tilde{P}_{\mathrm{ini},\xi} := \frac{1}{kD}\sum_{l,l'=0}^{N-1}\tilde{\Gamma}_{\xi l}^c(\mathbf{M}^{-1})_{ll'}P_{l'}(0).$$

The second term in the right-hand side describes the coupling between body modes and the first term describes the coupling mediated by the surface waves. This gives

$$\tilde{P}_\xi(x) = \left(\exp\left(\tilde{\mathfrak{M}}\frac{x}{kD}\right)\tilde{P}_{\mathrm{ini}}\right)_\xi, \tag{55}$$

with the kernel

$$\tilde{\mathfrak{M}}_{\xi\xi'} = \sum_{l,l'=0}^{N-1}\tilde{\Gamma}_{\xi l}^c(\mathbf{M}^{-1})_{ll'}\tilde{\Gamma}_{l'\xi'}^c\tilde{\mathcal{N}}(\xi') + \tilde{\Gamma}_{\xi\xi'}^c\tilde{\mathcal{N}}(\xi') - \tilde{\Lambda}_\xi^c\delta(\xi'-\xi). \tag{56}$$

Fifth, the equipartition principle takes, here, the following form: As $x \to +\infty$, $P_j(x)$ and $\tilde{P}_\xi(x)$ converge to

$$\mathcal{P}_\infty(\omega) = \frac{\mathcal{P}(0;\omega)}{kD\displaystyle\int_0^\infty\tilde{\mathcal{N}}(\xi)d\xi}, \tag{57}$$

which is here equal to

$$\mathcal{P}_\infty(\omega) = \frac{\pi \mathcal{P}(0;\omega)}{n_1 k D},$$

by (36). This means that most of the power is carried by body modes (the fraction of power carried by the surface modes is of the order of d/D). However, since the spatial profiles of the body waves extend throughout $(0,D)$ while those of the surface waves are concentrated on $(0,d)$, the contribution of the body waves and of the surface waves to the Wigner transform (14) of the normal derivative of the field at the surface are of the same order, and we have

$$\int_{-\infty}^{+\infty} W_D^{\rm s}(x,\kappa;t,\omega)dt \simeq \mathcal{P}_\infty(\omega) \sum_{j=0}^{N(\omega)-1} \frac{\partial_z \phi_j(0;\omega)^2}{\beta_j(\omega)} \delta\big(\kappa - \beta_j(\omega)\big) + \mathcal{P}_\infty(\omega) \frac{\partial_z \tilde{\phi}_\xi(0;\omega)^2}{k\xi} \tilde{\mathcal{N}}(\xi)\big|_{\xi=\kappa/k}, \tag{58}$$

for propagation distances x much larger than $D/(k\sigma^2 \ell_c)$.

Sixth, if we assume a delta-correlated model,

$$\mathbb{E}[\nu(x,z)\nu(x',z')] = \sigma^2 \mathbf{1}_{[0,d']}(z)\delta(x-x')\delta(z-z'), \tag{59}$$

for some d' larger than d, then the differential scattering cross sections take the simplified forms,

$$\begin{aligned}
\Gamma^c_{jl}(\omega) =& \frac{k^4}{4\beta_j \beta_l(\omega)} \sigma^2 \int_0^\infty \phi_j^2 \phi_l^2(z;\omega) dz, \\
\tilde{\Gamma}^c_{j\xi}(\omega) =& \frac{k^3}{4\beta_j(\omega)\xi} \sigma^2 \int_0^\infty \phi_j^2 \tilde{\phi}_\xi^2(z;\omega) dz, \\
\tilde{\Gamma}^c_{\xi\xi'}(\omega) =& \frac{k^2}{4\xi\xi'} \sigma^2 \left[\int_0^\infty \tilde{\phi}_\xi^2 \tilde{\phi}_{\xi'}^2(z;\omega) dz + k^2 D_0 \right].
\end{aligned}$$

4. Propagation in a thick scattering waveguide with a complex background

Let $D > d$. We consider the problem

$$\Delta \hat{u} + k^2 n_D^2(x,z)\hat{u} = -\hat{f}(x,z;\omega), \quad (x,z) \in \mathbb{R} \times (0,D), \tag{60}$$

with $k = \omega/c_o$, Dirichlet boundary condition at $z=0$ and Neumann boundary condition at $z = D$, and

$$n_D^2(x,z) = {\rm n}_{{\rm b},D}^2(z) + \varepsilon \nu(x,z) + \varepsilon \frac{\sqrt{D_0}}{\sqrt{D}} \mu(x,z) \text{ if } z \in (0,D), \tag{61}$$

where

- the function ${\rm n}_{{\rm b},D}(z)$ is such that

$$ {\rm n}_{{\rm b},D}(z) = \begin{cases} {\rm n_b}(z) & \text{if } z \in [0,d], \\ \tilde{\rm n}_{\rm b}((z-d)/(D-d)) & \text{if } z \in (d,D], \end{cases} \tag{62}$$

  where ${\rm n_b}(z)$ is non-increasing on $[0,d]$ from n_0 to n_1, with ${\rm n_b}(d) = n_1$, and $\tilde{\rm n}_{\rm b}(\zeta)$ is non-increasing on $[0,1]$ from n_1 to n_2. Futhermore, $n_0 > n_1 \geq n_2$.
- ν is a zero-mean random process, stationary in x, compactly supported in z, so that $\mathbb{E}[\nu(x,z)\nu(x',z')] = 0$ as soon as $\max(z,z') > d'$ for some $d' > 0$.
- μ is a zero-mean random process, stationary in x and z, independent from ν. D_0 is a reference length that is added so that μ is dimensionless. We denote

$$\mathbb{E}[\mu(x,z)\mu(x',z')] = \mathcal{R}^\mu(x-x', z-z').$$

4.1. Piecewise constant background

In this subsection we consider the case where the function $\tilde{\mathrm{n}}_{\mathrm{b}}(\zeta)$ is piecewise constant and of the form

$$\tilde{\mathrm{n}}_{\mathrm{b}}(\zeta) = \begin{cases} n_1 & \text{if } \zeta \in (0,\alpha], \\ n_2 & \text{if } \zeta \in (\alpha,1], \end{cases} \tag{63}$$

which means that

$$\mathrm{n}_{\mathrm{b},D}(z) = \begin{cases} \mathrm{n}_{\mathrm{b}}(z) & \text{if } z \in [0,d], \\ n_1 & \text{if } z \in (d, d+\alpha(D-d)], \\ n_2 & \text{if } z \in (d+\alpha(D-d), D]. \end{cases} \tag{64}$$

We have

Proposition 4.1. *Proposition 3.2 holds true with model (64) provided that the expressions for the density of states $\mathcal{N}$, the group velocities v_ξ, and the correlation function $\tilde{\mathcal{R}}^\mu$ are updated as follows. The normalized density of states is*

$$\mathcal{N}(g) = \frac{\alpha}{2\pi\sqrt{n_1^2 - g}} \mathbf{1}_{(0,n_1^2)}(g) + \frac{1-\alpha}{2\pi\sqrt{n_2^2-g}} \mathbf{1}_{(0,n_2^2)}(g), \tag{65}$$

and

$$\tilde{\mathcal{N}}(\xi) = 2\xi\mathcal{N}(\xi^2) = \frac{\alpha\xi}{\pi\sqrt{n_1^2-\xi^2}} \mathbf{1}_{(0,n_1)}(\xi) + \frac{(1-\alpha)\xi}{\pi\sqrt{n_2^2-\xi^2}} \mathbf{1}_{(0,n_2)}(\xi). \tag{66}$$

The group velocities are

$$v_\xi = \frac{c_o\xi}{n_1^2} \mathbf{1}_{(n_2,n_1)}(\xi) + c_o\xi \frac{\alpha\sqrt{n_2^2-\xi^2} + (1-\alpha)\sqrt{n_1^2-\xi^2}}{\alpha n_1^2\sqrt{n_2^2-\xi^2} + (1-\alpha)n_2^2\sqrt{n_1^2-\xi^2}} \mathbf{1}_{(0,n_2)}(\xi). \tag{67}$$

For $\xi, \xi' \in (0, n_1)$, the correlation function $\tilde{\mathcal{R}}^\mu_{\xi\xi'}$ is given by

$$\begin{aligned} \tilde{\mathcal{R}}^\mu_{\xi\xi'}(x;\omega) = \frac{k^2 D_0}{(\alpha+\tilde{\alpha}_\xi)(\alpha+\tilde{\alpha}_{\xi'})} \int_0^\infty \mathcal{R}^\mu(x,z) \Big[\alpha^2 \cos\big(kz\sqrt{n_1^2-\xi^2}\big) \cos\big(kz\sqrt{n_1^2-\xi'^2}\big) \\ + \tilde{\alpha}_\xi\tilde{\alpha}_{\xi'} \cos\big(kz\sqrt{n_2^2-\xi^2}\big)\cos\big(kz\sqrt{n_2^2-\xi'^2}\big)\Big] dz, \end{aligned} \tag{68}$$

with

$$\tilde{\alpha}_\xi = \frac{1-\alpha}{2} \frac{(n_1^2+n_2^2) - 2\xi^2}{n_2^2-\xi^2} \mathbf{1}_{(0,n_2)}(\xi). \tag{69}$$

The other quantities (group velocities, etc) are unchanged.

Proof. We essentially need to revisit the calculations of the density of states and the normalizing constants. See Appendix B.3. □

The discussion following Proposition 3.2 is still valid. The only difference is that the equipartioned power (57) is here equal to

$$\mathcal{P}_\infty(\omega) = \frac{\pi\mathcal{P}(0;\omega)}{[\alpha n_1 + (1-\alpha)n_2]kD},$$

by (66).

4.2. Smooth background

In this subsection we consider the case where the function $\tilde{\mathrm{n}}_{\mathrm{b}}(\zeta)$ is smooth and non-increasing on $[0,1]$ from n_1 to n_2. We have

Proposition 4.2. *Proposition 3.2 holds true with model (63) provided that the expressions for the density of states $\mathcal{N}$, the group velocities v_ξ, and the correlation function $\tilde{\mathcal{R}}^\mu$ are updated as follows. The normalized density of states is*

$$\mathcal{N}(g) = \frac{1}{2\pi}\int_0^1 \frac{1}{\sqrt{\tilde{\mathrm{n}}_{\mathrm{b}}^2(\zeta) - g}} d\zeta \tag{70}$$

and

$$\tilde{\mathcal{N}}(\xi) = 2\xi\mathcal{N}(\xi^2). \tag{71}$$

The group velocities are

$$\frac{1}{v_\xi} = \frac{1}{c_o}\left(\xi - \frac{\mathcal{M}(\xi^2)}{2\xi\mathcal{N}(\xi^2)}\right), \tag{72}$$

with

$$\mathcal{M}(g) = \frac{1}{\pi}\int_0^1 \sqrt{\tilde{\mathrm{n}}_{\mathrm{b}}^2(\zeta) - g}d\zeta. \tag{73}$$

For $\xi, \xi' \in (0, n_1)$, the correlation function $\tilde{\mathcal{R}}^\mu_{\xi\xi'}$ is given by

$$\begin{aligned}\tilde{\mathcal{R}}^\mu_{\xi\xi'}(x;\omega) =& k^2 D_0 \int_0^\infty \mathcal{R}^\mu(x,z)C_{\xi\xi'}(z;\omega)dz,\\ C_{\xi\xi'}(z;\omega) =& \frac{\displaystyle\int_0^1 \tilde{\rho}_\xi(\zeta)\tilde{\rho}_{\xi'}(\zeta)\cos\left(kz\sqrt{\tilde{\mathrm{n}}_{\mathrm{b}}^2(\zeta)-\xi^2}\right)\cos\left(kz\sqrt{\tilde{\mathrm{n}}_{\mathrm{b}}^2(\zeta)-\xi'^2}\right)d\zeta}{\displaystyle\int_0^1 \tilde{\rho}_\xi(\zeta)d\zeta \int_0^1 \tilde{\rho}_{\xi'}(\zeta)d\zeta},\end{aligned} \tag{74}$$

with

$$\tilde{\rho}_\xi(\zeta) = \frac{1}{\sqrt{\tilde{\mathrm{n}}_{\mathrm{b}}^2(\zeta) - \xi^2}}\mathbf{1}_{(0,\tilde{\mathrm{n}}_{\mathrm{b}}(\zeta))}(\xi). \tag{75}$$

We note that, when the process μ is delta-correlated as in (59), we simply have

$$\tilde{\mathcal{R}}^\mu_{\xi\xi'}(x;\omega) = \frac{k^2 D_0 \sigma^2}{2}\frac{\displaystyle\int_0^1 \tilde{\rho}_\xi(\zeta)\tilde{\rho}_{\xi'}(\zeta)d\zeta}{\displaystyle\int_0^1 \tilde{\rho}_\xi(\zeta)d\zeta\int_0^1 \tilde{\rho}_{\xi'}(\zeta)d\zeta}.$$

Proof. We essentially need to revisit the calculations of the density of states and the normalizing constants. We present the details in Appendix B.4. □

The discussion following Proposition 3.2 is still valid. The only difference is that the equipartioned power (57) is here equal to

$$\mathcal{P}_\infty(\omega) = \frac{\pi\mathcal{P}(0;\omega)}{kD\displaystyle\int_0^1 \tilde{\mathrm{n}}_{\mathrm{b}}(\zeta)d\zeta},$$

by (70)-(71).

5. Numerical illustrations

In this section we illustrate some of the above results with numerical simulations. The model that we discuss corresponds to the one set forth in Subsection 3.1 with a finite value for the waveguide thickness parameter D. In particular, we illustrate how the smooth background profile for the index of refraction affects the spectrum associated with body and surface modes, and explore how the coupling via the term involving the kernel coefficients Γ^c_{jl} in Proposition 3.1 affects the power distribution over the waveguide modes.

In Figure 1 above, we show the configuration that we consider: the left plot gives the geometry of the waveguide and the two right plots give two realizations of the randomly perturbed index of refraction. The index of refraction is relatively large for $z < d$ which implies the existence of a set of surface modes. The index of refraction comprises a smooth, and decreasing in z, background component and a random component as exemplified by the two right plots. In Figure 2, we depict the modes and the associated set of eigenvalues evaluated in the background profile. The red stars correspond to the surface modes. These are indeed concentrated in $0 \le z < d$. The bottom left plot shows that the body modes are oscillatory across the entire composite waveguide as expected.

In Figure 3, left, we plot the phase and group velocities for the modes associated with the background profile for the index of refraction. We note how these reflect the presence of the section supporting the surface modes. The random medium fluctuations imply the coupling of the modes, which is illustrated in Figure 3, right. Here, we show the coupling coefficients to the other modes. The top plot shows the coupling to surface mode as function of mode number (for both surface and body modes), while the bottom plot shows the coupling to body modes. That is, in the top plot, we depict

$$\tilde{\Gamma}^{\rm s}_j = \sum_{l \in \mathcal{I}^{\rm s},\ l \neq j} \Gamma^c_{jl},$$

for $\mathcal{I}^{\rm s}$ denoting the set of surface modes, and in the bottom plot

$$\tilde{\Gamma}^{\rm b}_j = \sum_{l \in \mathcal{I}^{\rm b},\ l \neq j} \Gamma^c_{jl},$$

for $\mathcal{I}^{\rm b}$ denoting the set of body modes.

In Figure 4, the top left plot shows the coupling coefficients Γ^c_{jl} as function of mode pair (j, l) with blue color corresponding to small values and red to large values. The strong coupling in between the surface modes is seen in the bottom left corner. The top right plot in the figure shows the evolution of mean mode powers toward equipartition. Initially, at $x = 0$, the power is carried by the surface modes and then there is an evolution where power is transferred to body modes due to the random medium fluctuations and such that the configuration approaches equipartition. This plot corresponds to the random medium fluctuations being supported in the section $z < d$ as in in Figure 1b. The two bottom plots in Figure 4 correspond to the top plots only that here the random medium is supported in the full section $z < D$ as in Figure 1c. Note that then the coupling in between the body modes becomes stronger.

In Figure 5, we depict the mean intensity carried by the modes in the case with random fluctuations. The top plot pertains to the intensity carried by the body modes, the middle plot to the intensity carried by the surface modes and the bottom plot to the total intensity. Again, we observe how the random fluctuations induce power (or intensity) transfer, here, from surface to body modes and an evolution towards equipartition. In the case with many more body than surface modes, induced by the value of D, most of the power transfers from the surface to the body modes. This happens even though the random medium fluctuations are supported only in the section $z < d$.

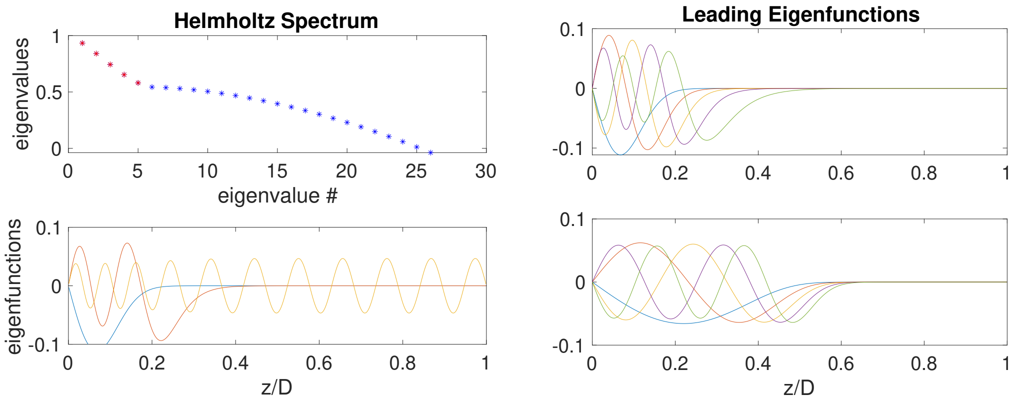

FIGURE 2. *Left:* The figure illustrates the spectrum of the one-dimensional Schrödinger operator associated with the background index of refraction profile when D is finite. The top plot shows the normalized eigenvalues $\gamma_i/(n_0 k)^2$. There are 5 surface modes indicated by the red stars. The bottom plot shows eigenfunctions 1, 4 and 20. The modes 1 and 4 correspond to surface modes and are supported essentially in the surface layer, while the mode 20 corresponds to a body mode and is oscillatory and is supported in the full waveguide. *Right:* The top plot shows the first 5 eigenfunctions corresponding to the background profile shown in Figure 1 when $d/D \approx .2$. The bottom plot shows the corresponding eigenfunctions when the width of the surface layer is tripled till $d/D \approx .6$ and we see that the support of the eigenfunctions have been correspondingly extended.

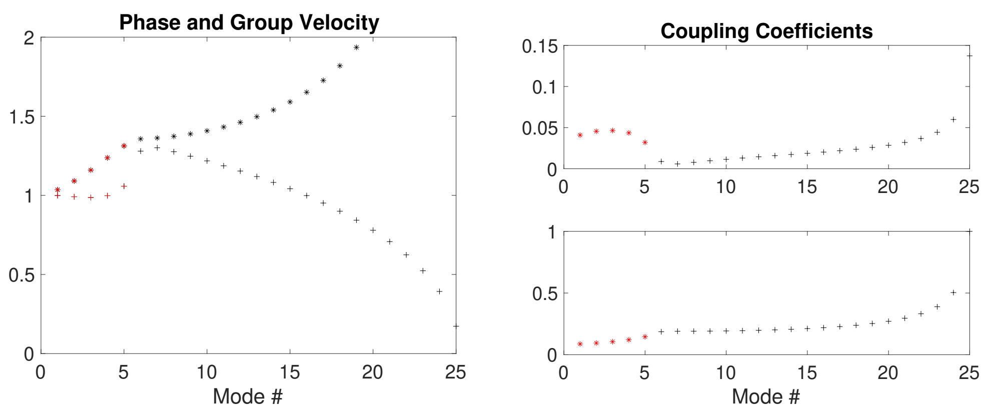

FIGURE 3. *Left:* The crosses in the figure show the relative mode group velocities $v_{j,D}(n_0/c_0)$, with the red crosses corresponding to surface modes and the black crosses the body modes. The stars give the corresponding phase velocities $v^{\rm p}_{j,D}(n_0/c_0) = n_0 k/\beta_{j,D}$. *Right:* The figure shows the coupling coefficients $\Lambda^c_{j,D}$ decomposed into coupling to respectively surface and body modes. The top plot shows the components attributable to loss to surface modes, as function of mode (with red stars corresponding to surface modes) and the bottom plot shows the components attributable to loss to body modes, again as function of mode. In this and the next figures we use a delta correlated model for the medium fluctuations.

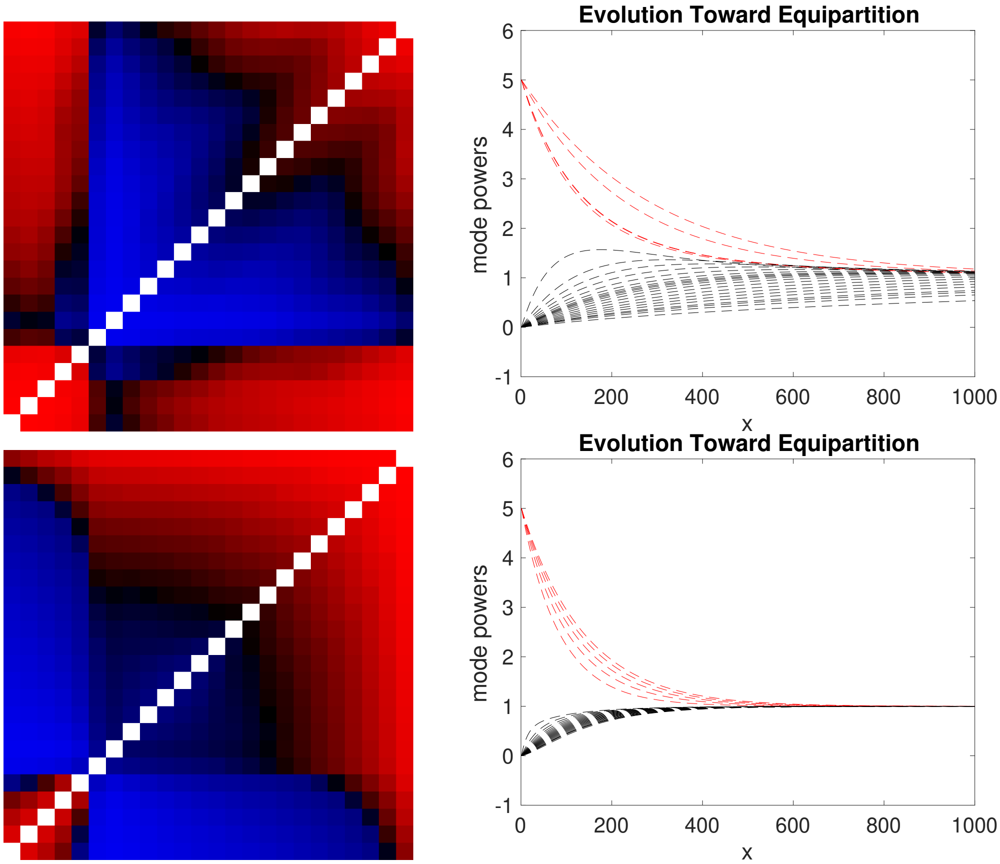

Figure 4. *Top left plot:* The figure shows how the values of the cross mode coupling coefficients Γ^c_{jl} depend on the mode indices (j, l). The first 5 modes are the surface modes and pictured in the left bottom corner. Here the random medium fluctuations are supported in the section $z < d$ (as in in Figure 1b) giving a relatively weak coupling in between body modes. *Top right plot:* The figure shows the evolution toward equipartition for the mean mode powers. The red dashed lines correspond to surface modes and the black dashed lines to body modes. The mode coupling, as shown in the left plot, gives an exchange of power in between the modes and evolution toward equipartition. *The bottom two plots:* correspond to the top plots except that the random medium fluctuations are now supported in the full waveguide (as in Figure 1c) giving stronger mode coupling among the body modes.

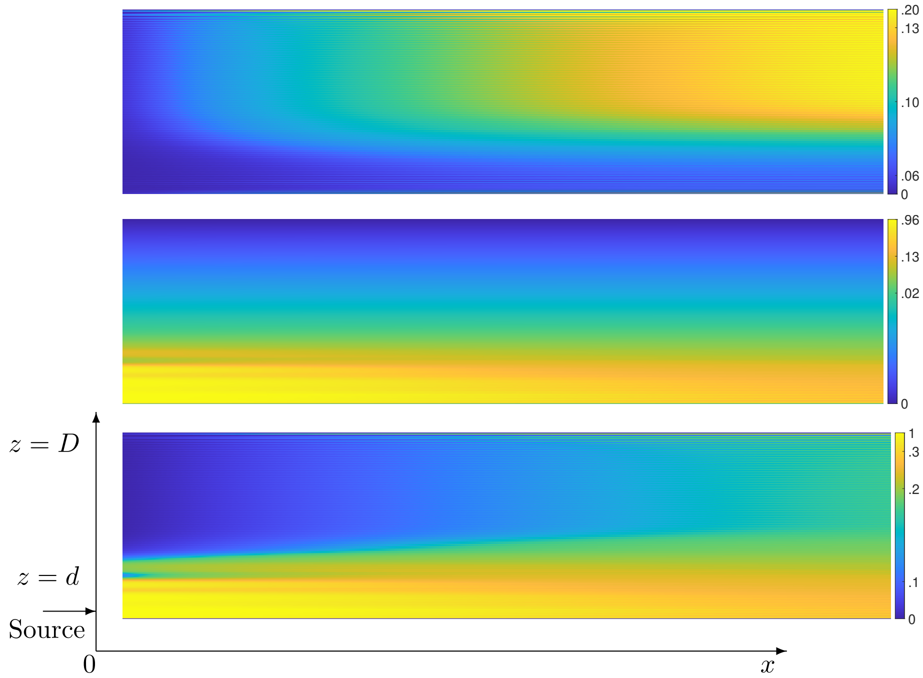

FIGURE 5. This figure illustrates the evolution of mode powers and intensities in the case with random medium fluctuations so that the index of refraction is as shown in Figure 1b. Thus, we incorporate clutter on top of the smooth background profile, however, with the random medium fluctuations being supported in the section $z < d$ only. Note that we used a nonlinear color scaling to enhance the transition zones. *The top plot:* shows the mean intensity of the body mode components which are supported in the full section $z < D$. It approaches a steady state configuration for large propagation, x, distances. *The middle plot:* shows the mean intensity of the surface mode components, supported essentially for $z < d$. For large propagation distance, x, it is seen that most of the initial power has been transferred to the body modes. *The bottom plot:* shows the total intensity distribution, the sum of the surface and body mode intensities shown in the two top plots, and illustrates the evolution to a steady state equipartition regime.

6. Conclusion

We considered the propagation scattering of waves in a randomly heterogeneous half-space that has a thin layer beneath the surface that supports a finite number of surface modes. We derived a novel system of radiative transfer equations that governs the evolution of the Wigner transform of the normal derivative of the wave field on the boundary. We analyzed the dynamics revealed by this system that couples the surface modes corresponding to the discrete spectrum and the body modes corresponding to the continuous spectrum which are determined by the smooth deterministic background. This mode coupling induces a non-trivial process that involves a slowly evolving metastable surface mode distribution and ultimately leads to energy equipartition between all modes. This implies, for example, that initially excited surface modes effectively loose energy as they propagate. These results pave the way to analyze the associated inverse problem, addressing the outstanding claim that the background index of refraction can be robustly determined from the mentioned Wigner transform or related albedo operator.

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Appendix A. The spectral problems

Here the function $z \in [0,+\infty) \mapsto \mathrm{n_b}(z)$ is such that $\mathrm{n_b}(0) = n_0$, $\mathrm{n_b}(z)$ is non-increasing on $[0,d]$ from n_0 to $n_1 < n_0$, and $\mathrm{n_b}(z) = n_1$ for $z \geq d$ (see Figure 1).

We first introduce the unnormalized functions ψ_γ which appear in the expressions of the eigenfunctions in the next subsections. For any $\gamma \in \mathbb{R}$ we denote by ψ_γ the unique solution of the second-order differential equation

$$(\partial_z^2 + k^2 \mathrm{n_b^2}(z))\psi_\gamma(z) = \gamma\psi_\gamma(z), \quad z \in (0,+\infty), \tag{76}$$

starting from $\psi_\gamma(z=0) = 0$ and $\partial_z\psi_\gamma(z=0) = 1$. The solution has the following form below the thin layer ($\mathrm{n_b}(z) = n_1$ for $z > d$):

- If $\gamma < n_1^2 k^2$ we have for $z > d$:

$$\psi_\gamma(z) = \psi_\gamma(d)\cos\big(\sqrt{n_1^2k^2-\gamma}(z-d)\big) + \frac{\partial_z\psi_\gamma(d)}{\sqrt{n_1^2k^2-\gamma}}\sin\big(\sqrt{n_1^2k^2-\gamma}(z-d)\big), \tag{77}$$

  and ψ_γ is bounded.
- If $\gamma = n_1^2 k^2$ we have for $z > d$:

$$\psi_\gamma(z) = \psi_\gamma(d) + \partial_z\psi_\gamma(d)(z-d),$$

  and ψ_γ is bounded if and only if $\partial_z\psi_\gamma(d) = 0$.
- If $\gamma > n_1^2 k^2$ we have for $z > d$:

$$\psi_\gamma(z) = \psi_\gamma(d)\cosh\big(\sqrt{\gamma-n_1^2k^2}(z-d)\big) + \frac{\partial_z\psi_\gamma(d)}{\sqrt{\gamma-n_1^2k^2}}\sinh\big(\sqrt{\gamma-n_1^2k^2}(z-d)\big), \tag{78}$$

  and ψ_γ is bounded if and only if

$$\psi_\gamma(d) + \frac{\partial_z\psi_\gamma(d)}{\sqrt{\gamma-n_1^2k^2}} = 0, \tag{79}$$

and then it belongs to $L^2(0,+\infty)$. There is a finite number N of values of γ for which this equation is satisfied, they are denoted by $\gamma_0 > \cdots > \gamma_{N-1}$.

A.1. A finite waveguide containing a thin layer

Let $D > d$. We consider the spectral problem associated to the one-dimensional Schrödinger operator $(\partial_z^2 + k^2 \mathrm{n}_\mathrm{b}^2(z))\phi(z) = \gamma\phi(z)$ with Dirichlet boundary condition at $z = 0$ and Neumann boundary condition at $z = D$. The spectrum is discrete. The eigenvalues are simple and denoted by $\gamma_{0,D} > \cdots > \gamma_{j,D} > \cdots$. Since $\mathrm{n}_\mathrm{b}(z) = n_1$ for $z \in (d, D)$ and the eigenfunctions satisfy the Neumann boundary condition at $z = D$, we have

$$\psi_{\gamma_{j,D}}(d) \tanh\left(\sqrt{\gamma_{j,D} - n_1^2 k^2}(D - d)\right) + \frac{\partial_z \psi_{\gamma_{j,D}}(d)}{\sqrt{\gamma_{j,D} - n_1^2 k^2}} = 0 \tag{80}$$

if $\gamma_{j,D} > n_1^2 k^2$ or

$$\psi_{\gamma_{j,D}}(d) \tan\left(\sqrt{n_1^2 k^2 - \gamma_{j,D}}(D - d)\right) - \frac{\partial_z \psi_{\gamma_{j,D}}(d)}{\sqrt{n_1^2 k^2 - \gamma_{j,D}}} = 0 \tag{81}$$

if $\gamma_{j,D} < n_1^2 k^2$. The normalized eigenfunctions have the form

$$\phi_{j,D}(z) = \sqrt{r_{j,D}}\psi_{\gamma_{j,D}}(z), \tag{82}$$

with

$$r_{j,D}^{-1} = \int_0^D \psi_{\gamma_{j,D}}(z)^2 dz. \tag{83}$$

By the Sturm-Liouville theory, if $v \in L^2(0,+\infty)$, we have

$$v(z) = \sum_{j=0}^{\infty} v_{j,D}\phi_{j,D}(z) \text{ in } (0, D),$$

where

$$v_{j,D} = \int_0^D v(z)\phi_{j,D}(z)dz.$$

This can also be written as

$$v(z) = \int_{\mathbb{R}} V_D(\gamma)\psi_\gamma(z)\rho_D(d\gamma) \quad \text{in } (0, D),$$

where

$$V_D(\gamma) = \int_0^D v(z)\psi_\gamma(z)dz$$

and

$$\rho_D(d\gamma) = \sum_{j=0}^{\infty} r_{j,D}\delta_{\gamma_{j,D}}(d\gamma).$$

By Chapter 9 in Ref. [9] we have $V_D \to V$ and $\rho_D \to \rho$ in appropriate topologies, where

$$V(\gamma) = \int_0^\infty v(z)\psi_\gamma(z)dz \tag{84}$$

and ρ is the measure described in the following proposition. We have for any $v \in L^2(0,+\infty)$:

$$v(z) = \int_{\mathbb{R}} V(\gamma)\psi_\gamma(z)\rho(d\gamma) \quad \text{in } (0,+\infty) \tag{85}$$

and the Parseval relation is satisfied:

$$\int_0^{+\infty} v(z)^2 dz = \int_{\mathbb{R}} V(\gamma)^2 \rho(d\gamma). \tag{86}$$

Proposition A.1. *1. For any $j < N$, $\gamma_{j,D} \to \gamma_j$ as $D \to +\infty$, where the γ_j's are the solutions in $(n_1^2k^2, +\infty)$ of (79). Furthermore,*

$$r_{j,D} \overset{D\to+\infty}{\longrightarrow} r_j := \frac{2\sqrt{\gamma_j - n_1^2k^2}}{2\sqrt{\gamma_j - n_1^2k^2}\int_0^d \psi_{\gamma_j}(z)^2 dz + \psi_{\gamma_j}(d)^2}. \tag{87}$$

2. The measure $\rho_D \to \rho$ as $D \to +\infty$ with

$$\rho(d\gamma) = \sum_{j=0}^{N-1} r_j \delta_{\gamma_j}(d\gamma) + r_\gamma \mathbf{1}_{(-\infty, n_1^2k^2)}(\gamma) d\gamma \tag{88}$$

and

$$r_\gamma = \frac{1}{\pi} \frac{\sqrt{n_1^2k^2 - \gamma}}{(n_1^2k^2 - \gamma)\psi_\gamma(d)^2 + \partial_z\psi_\gamma(d)^2}. \tag{89}$$

Proof. 1. The eigenvalues $\gamma_{j,D}$ that belong to $(n_1^2k^2, +\infty)$ satisfy (80). The left-hand side of (80) converges uniformly in γ as $D \to +\infty$ to the left-hand side of (79), so the $\gamma_{j,D}$ converge to the zeroes of (79), here denoted by γ_j. Moreover, $r_{j,D}$ is defined by (83) and it converges to r_j defined by (87) as $D \to +\infty$ by (78). This completes the proof of the first statement.

2. If g is a test function supported in $(n_1^2k^2, +\infty)$, we have for D large enough

$$\int_{\mathbb{R}} g(\gamma)\rho_D(d\gamma) = \sum_{j<N} r_{j,D}\, g(\gamma_{j,D}),$$

where N is the number of discrete eigenvalues of the half-space problem (see section 2.1). The first statement of the proposition then gives

$$\int_{\mathbb{R}} g(\gamma)\rho_D(d\gamma) \overset{D\to+\infty}{\longrightarrow} \sum_{j<N} r_j g(\gamma_j).$$

From (81), for $j \geq N$ we find that $\sqrt{n_1^2k^2 - \gamma_{j,D}}(D-d) \to (j-N+1)\pi$ as $D \to +\infty$. Therefore $\gamma_{j,D} \simeq n_1^2k^2 - (j-N+1)^2\pi^2/(D-d)^2$ and

$$\gamma_{j+1,D} - \gamma_{j,D} \simeq -\frac{2\pi}{D-d}\sqrt{n_1^2k^2 - \gamma_{j,D}}.$$

This shows that the density of eigenvalues at $\gamma \in (0, n_1^2k^2)$ is

$$\mathcal{N}_D(\gamma) = \frac{D}{2\pi\sqrt{n_1^2k^2 - \gamma}},$$

that is, the number of eigenvalues $\gamma_{j,D}$ in $(\gamma, \gamma + \delta\gamma)$ is $\mathcal{N}_D(\gamma)\delta\gamma + o(\delta\gamma)$. This result can also be established by using the Weyl law for the Schrödinger operator: When $D \to +\infty$, the number $M_D(\gamma)$ of eigenvalues $\gamma_{j,D}$ larger than γ is

$$M_D(\gamma) = \frac{D}{2\pi}\int_{n_1^2k^2 - \kappa^2 > \gamma} d\kappa\big(1 + o(kD)\big) = \frac{D}{\pi}\sqrt{n_1^2k^2 - \gamma}\big(1 + o(kD)\big),$$

hence the density of eigenvalues is $\mathcal{N}_D(\gamma) = |\partial_\gamma M_D(\gamma)|$.

Moreover, if j_D is such that $\gamma_{j_D,D} \to \gamma$ as $D \to +\infty$, then by (77):

$$\frac{1}{D r_{j_D,D}} = \frac{1}{D}\int_0^D \psi_{\gamma_{j_D,D}}(z)^2 dz \overset{D\to+\infty}{\longrightarrow} \frac{1}{2}\psi_\gamma(d)^2 + \frac{1}{2}\frac{\partial_z\psi_\gamma(d)^2}{n_1^2k^2 - \gamma}.$$

By combining these results, if g is a test function supported in $(-\infty, n_1^2k^2)$, we have for large D

$$\begin{aligned}\int_{\mathbb{R}} g(\gamma)\rho_D(d\gamma) &= \sum_{j\geq N} r_{j,D}\, g(\gamma_{j,D}) \\ &\overset{D\to+\infty}{\longrightarrow} \int_{-\infty}^{n_1^2k^2} \Big[\frac{1}{2}\psi_\gamma(d)^2 + \frac{1}{2}\frac{\partial_z\psi_\gamma(d)^2}{n_1^2k^2 - \gamma}\Big]^{-1} \frac{1}{2\pi\sqrt{n_1^2k^2 - \gamma}} g(\gamma) d\gamma,\end{aligned}$$

which completes the proof of the second statement. □

A.2. A half-space with a thin layer

We consider the spectral problem associated to the one-dimensional Schrödinger operator $(\partial_z^2 + k^2 \mathrm{n}_\mathrm{b}^2(z))\phi(z) = \gamma\phi(z)$ in $(0,+\infty)$ with Dirichlet boundary condition at $z = 0$. This problem can be studied as the limit as $D \to +\infty$ of the problem addressed in the previous subsection. The spectrum has a discrete and a continuous part.

Discrete spectrum. The eigenvalues are $\gamma_0 > \cdots > \gamma_{N-1}$ that are the solutions in $(n_1^2 k^2, +\infty)$ of (79). The associated eigenfunctions ϕ_j are defined by

$$\phi_j(z) = \sqrt{r_j}\psi_{\gamma_j}(z), \tag{90}$$

where r_j is defined by (87) and ψ_γ is defined by (76).

Continuous spectrum. The spectrum has a continuous part $\gamma \in (-\infty, n_1^2 k^2)$. The generalized eigenfunctions are

$$\phi_\gamma(z) = \sqrt{r_\gamma}\psi_\gamma(z), \tag{91}$$

where r_γ is defined by (89) and ψ_γ is defined by (76).

By (85), for any $v \in L^2(0,+\infty)$ we have

$$v(z) = \sum_{j=0}^{N-1} v_j \phi_j(z) + \int_{-\infty}^{n_1^2 k^2} v_\gamma \phi_\gamma(z) d\gamma \quad \text{in } (0,+\infty), \tag{92}$$

with

$$v_j = \int_0^{+\infty} v(z)\phi_j(z)dz, \qquad v_\gamma = \int_0^{+\infty} v(z)\phi_\gamma(z)dz, \tag{93}$$

and we have the following Parseval relation

$$\int_0^{+\infty} v(z)^2 dz = \sum_{j=0}^{N-1} v_j^2 + \int_{-\infty}^{n_1^2 k^2} v_\gamma^2 d\gamma. \tag{94}$$

Appendix B. Proofs of propositions

B.1. Proof of Proposition 2.1

We consider that the medium is randomly perturbed for $x \in (0, L^{(\varepsilon)})$, with $L^{(\varepsilon)} = L/\varepsilon^2$. For a fixed frequency ω, we expand the wave field as in (6):

$$\hat{u}(x,z) = \sum_{j=0}^{N-1} \hat{u}_j(x)\phi_j(z) + \int_{-\infty}^{n_1^2 k^2} \hat{u}_\gamma(x)\phi_\gamma(z) d\gamma.$$

The complex mode amplitudes satisfy the coupled equations for $x \in (0, L^{(\varepsilon)})$,

$$\partial_x^2 \hat{u}_j + \beta_j^2 \hat{u}_j = -\varepsilon k^2 \sum_{l=0}^{N-1} C_{j,l}(x)\hat{u}_l - \varepsilon k^2 \int_{-\infty}^{n_1^2 k^2} C_{j,\gamma'}(x)\hat{p}_{\gamma'} d\gamma', \tag{95}$$

for $j = 0, \ldots, N-1$,

$$\partial_x^2 \hat{u}_\gamma + \gamma \hat{u}_\gamma = -\varepsilon k^2 \sum_{l=0}^{N-1} C_{\gamma,l}(x)\hat{u}_l - \varepsilon k^2 \int_{-\infty}^{n_1^2 k^2} C_{\gamma,\gamma'}(x)\hat{u}_{\gamma'} d\gamma', \tag{96}$$

for $\gamma \in (-\infty, n_1^2 k^2)$, where

$$C_{j,l}(x) = (\phi_j, \phi_l \nu(x,\cdot))_{L^2}, \qquad C_{j,\gamma'}(x) = (\phi_j, \phi_{\gamma'} \nu(x,\cdot))_{L^2},$$
$$C_{\gamma,l}(x) = (\phi_\gamma, \phi_l \nu(x,\cdot))_{L^2}, \qquad C_{\gamma,\gamma'}(x) = (\phi_\gamma, \phi_{\gamma'} \nu(x,\cdot))_{L^2}$$

and $(\cdot,\cdot)_{L^2}$ stands for the standard scalar product in $L^2(0,+\infty)$.

We introduce the generalized forward-going and backward-going mode amplitudes,

$$\{a_j(x),\, b_j(x),\ j=0,\dots,N-1\} \text{ and } \{a_\gamma(x),\, b_\gamma(x),\ \gamma\in(0,n_1^2k^2)\}, \tag{97}$$

which are defined such that

$$\begin{aligned}\hat u_j(x) =& \frac{1}{\sqrt{\beta_j}}\Big(a_j(x)e^{i\beta_j x}+b_j(x)e^{-i\beta_j x}\Big),\\ \partial_x \hat u_j(z) =& i\sqrt{\beta_j}\Big(a_j(x)e^{i\beta_j x}-b_j(x)e^{-i\beta_j x}\Big),\quad j=0,\dots,N-1,\end{aligned}\tag{98}$$

and

$$\begin{aligned}\hat u_\gamma(x) =& \frac{1}{\gamma^{1/4}}\Big(a_\gamma(x)e^{i\sqrt{\gamma}x}+b_\gamma(x)e^{-i\sqrt{\gamma}x}\Big),\\ \partial_x \hat u_\gamma(x) =& i\gamma^{1/4}\Big(a_\gamma(x)e^{i\sqrt{\gamma}x}-b_\gamma(x)e^{-i\sqrt{\gamma}x}\Big),\quad \gamma\in(0,n_1^2k^2).\end{aligned}\tag{99}$$

We then substitute (98)-(99) into (95)-(96) in order to obtain the coupled system of random differential equations satisfied by the mode amplitudes in (97),

$$\begin{aligned}\partial_x a_j(x) =& \frac{i\varepsilon k^2}{2}\sum_{l'=0}^{N-1}\frac{C_{j,l'}(x)}{\sqrt{\beta_{l'}\beta_j}}\Big[a_{l'}(x)e^{i(\beta_{l'}-\beta_j)x}+b_{l'}(x)e^{i(-\beta_{l'}-\beta_j)x}\Big]\\ &+\frac{i\varepsilon k^2}{2}\int_0^{n_1^2k^2}\frac{C_{j,\gamma'}(x)}{\sqrt[4]{\gamma'}\sqrt{\beta_j}}\Big[a_{\gamma'}(x)e^{i(\sqrt{\gamma'}-\beta_j)x}+b_{\gamma'}(x)e^{i(-\sqrt{\gamma'}-\beta_j)x}\Big]d\gamma'\\ &+\frac{i\varepsilon k^2}{2}\int_{-\infty}^0\frac{C_{j,\gamma'}(x)}{\sqrt{\beta_j}}\hat u_{\gamma'}(x)e^{-i\beta_j x}d\gamma',\end{aligned}\tag{100}$$

$$\begin{aligned}\partial_x a_\gamma(x) =& \frac{i\varepsilon k^2}{2}\sum_{l'=0}^{N-1}\frac{C_{\gamma,l'}(x)}{\sqrt[4]{\gamma}\sqrt{\beta_{l'}}}\Big[a_{l'}(x)e^{i(\beta_{l'}-\sqrt{\gamma})x}+b_{l'}(x)e^{i(-\beta_{l'}-\sqrt{\gamma})x}\Big]\\ &+\frac{i\varepsilon k^2}{2}\int_0^{n_1^2k^2}\frac{C_{\gamma,\gamma'}(x)}{\sqrt[4]{\gamma'\gamma}}\Big[a_{\gamma'}(x)e^{i(\sqrt{\gamma'}-\sqrt{\gamma})x}+b_{\gamma'}(x)e^{i(-\sqrt{\gamma'}-\sqrt{\gamma})x}\Big]d\gamma'\\ &+\frac{i\varepsilon k^2}{2}\int_{-\infty}^0\frac{C_{\gamma,\gamma'}(x)}{\sqrt[4]{\gamma}}\hat u_{\gamma'}(x)e^{-i\sqrt{\gamma}x}d\gamma'.\end{aligned}\tag{101}$$

We have similar equations for b_j and b_γ. This system is complemented with the boundary conditions at $x=0$ and $x=L^{(\varepsilon)}$:

$$a_j(0)=a_{j,\mathrm{s}},\quad b_j(L^{(\varepsilon)})=0,\quad a_\gamma(0)=a_{\gamma,\mathrm{s}},\quad b_\gamma(L^{(\varepsilon)})=0,$$

where $a_{j,\mathrm{s}}$ and $a_{\gamma,\mathrm{s}}$ are defined by (10)-(11). The evanescent mode amplitudes $\hat u_\gamma$, $\gamma\in(-\infty,0)$, satisfy (96). By following the usual steps of the diffusion approximation theory set forth in [10, Chapter 20], we can prove the following

Proposition B.1. *Assume that*

$$\frac{1}{2\pi}\int e^{-iht}a_{j,\mathrm{s}}\Big(\omega+\frac{\varepsilon^2h}{2}\Big)\overline{a_{l,\mathrm{s}}\Big(\omega-\frac{\varepsilon^2h}{2}\Big)}dh \overset{\varepsilon\to0}{\longrightarrow} W^c_{jl,\mathrm{s}}(t,\omega),$$

which is equal to $a_{j,\mathrm{s}}(\omega)\overline{a_{l,\mathrm{s}}(\omega)}\delta(t)$ *if the source does not depend on* ε. *Then*

$$\frac{1}{2\pi}\int_{\mathbb{R}}e^{-iht}\mathbb{E}\left[a_j\Big(\frac{x}{\varepsilon^2},\omega+\frac{\varepsilon^2h}{2}\Big)\overline{a_l\Big(\frac{x}{\varepsilon^2},\omega-\frac{\varepsilon^2h}{2}\Big)}\right]dh \overset{\varepsilon\to0}{\longrightarrow} W^c_{jl,\mathrm{s}}(t,\omega)e^{-Q_{jl}(\omega)x},\quad j\neq l, \tag{102}$$

$$\frac{1}{2\pi}\int_{\mathbb{R}}e^{-iht}\mathbb{E}\left[a_j\Big(\frac{x}{\varepsilon^2},\omega+\frac{\varepsilon^2h}{2}\Big)\overline{a_j\Big(\frac{x}{\varepsilon^2},\omega-\frac{\varepsilon^2h}{2}\Big)}\right]dh \overset{\varepsilon\to0}{\longrightarrow} W^c_j(x;t+x/v_j,\omega), \tag{103}$$

where

$$\mathrm{Re}\big(Q_{jl}(\omega)\big) = \frac{\Lambda_j^s(\omega)+\Lambda_l^s(\omega)}{2} + \frac{\Gamma_{jj}^1(\omega)+\Gamma_{ll}^1(\omega)-2\Gamma_{jl}^1(\omega)}{2},$$
$$\Gamma_{jl}^1(\omega) = \frac{k^4}{4\beta_j\beta_l(\omega)}\int_0^\infty \mathbb{E}\big[C_{j,j}(0;\omega)C_{l,l}(x;\omega)\big]+\mathbb{E}\big[C_{l,l}(0;\omega)C_{j,j}(x;\omega)\big]dx,$$

$\Lambda_j^s(\omega)$ is defined by (21), and the $W_j^c(x;t,\omega)$ satisfy

$$\partial_x W_j^c + \frac{1}{v_j}\partial_t W_j^c = \sum_{l\neq j}\Gamma_{jl}^c W_l^c - \Lambda_j^s W_j^c,$$

starting from $W_j^c(x=0;t,\omega)=W_{jj,s}^c(t,\omega)$.

Proof. As shown in [10, Chapter 20], the forward scattering approximation is valid in our scaling regime, that is to say, we can make the approximation $b_j \simeq 0$ and $b_\gamma \simeq 0$, and the coupling with the evanescent modes only gives rise to an effective deterministic phase modulation that we will not take into account here. The wave mode amplitudes then satisfy the simplified system

$$\partial_x a_j(x) = \frac{i\varepsilon k^2}{2}\sum_{l'=0}^{N-1}\frac{C_{j,l'}(x)}{\sqrt{\beta_{l'}\beta_j}}a_{l'}(x)e^{i(\beta_{l'}-\beta_j)x} + \frac{i\varepsilon k^2}{2}\int_0^{n_1^2k^2}\frac{C_{j,\gamma'}(x)}{\sqrt[4]{\gamma'}\sqrt{\beta_j}}a_{\gamma'}(x)e^{i(\sqrt{\gamma'}-\beta_j)x}d\gamma', \qquad (104)$$

and a similar equation holds for a_γ. We denote

$$U_{j,l}^\varepsilon(x;\omega,h) = a_j\left(\frac{x}{\varepsilon^2},\omega+\frac{\varepsilon^2 h}{2}\right)\overline{a_l\left(\frac{x}{\varepsilon^2},\omega-\frac{\varepsilon^2 h}{2}\right)},$$

and similarly for $U_{j,\gamma}^\varepsilon$, etc. By expanding $\beta_j(\omega\pm\varepsilon^2 h/2)$ at ω, we get that $U_{j,l}^\varepsilon$ satisfies

$$\begin{aligned}\frac{dU_{j,l}^\varepsilon}{dx} =& \frac{ik^2}{2\varepsilon}\sum_{l'=0}^{N-1}\frac{C_{j,l'}(\frac{x}{\varepsilon^2})}{\sqrt{\beta_{l'}\beta_j}}U_{l',l}^\varepsilon e^{i(\beta_{l'}-\beta_j)\frac{x}{\varepsilon^2}}e^{i(\beta_{l'}'-\beta_j')\frac{xh}{2}} - \frac{ik^2}{2\varepsilon}\sum_{l'=0}^{N-1}\frac{C_{l,l'}(\frac{x}{\varepsilon^2})}{\sqrt{\beta_{l'}\beta_l}}U_{j,l'}^\varepsilon e^{i(\beta_l-\beta_{l'})\frac{x}{\varepsilon^2}}e^{i(\beta_{l'}'-\beta_l')\frac{xh}{2}}\\ &+\frac{ik^2}{2\varepsilon}\int_0^{n_1^2k^2}\frac{C_{j,\gamma'}(\frac{x}{\varepsilon^2})}{\sqrt[4]{\gamma'}\sqrt{\beta_j}}U_{\gamma',l}^\varepsilon e^{i(\sqrt{\gamma'}-\beta_j)\frac{x}{\varepsilon^2}}d\gamma' e^{-i\beta_j'\frac{xh}{2}}\\ &-\frac{ik^2}{2\varepsilon}\int_0^{n_1^2k^2}\frac{C_{l,\gamma'}(\frac{x}{\varepsilon^2})}{\sqrt[4]{\gamma'}\sqrt{\beta_l}}U_{\gamma',l}^\varepsilon e^{-i(\sqrt{\gamma'}-\beta_l)\frac{x}{\varepsilon^2}}d\gamma' e^{-i\beta_l'\frac{xh}{2}},\end{aligned}$$

and we get similar equations for $U_{j,\gamma}^\varepsilon$, etc. We introduce

$$V_{j,l}^\varepsilon(x;\omega,t) = \frac{1}{2\pi}\int e^{-ih(t-(\beta_j'(\omega)+\beta_l'(\omega))\frac{x}{2})}U_{j,l}^\varepsilon(x;\omega,h)dh,$$

and similarly for $V_{j,\gamma}^\varepsilon$, etc. It satisfies

$$\begin{aligned}\frac{\partial V_{j,l}^\varepsilon}{\partial x} + \frac{\beta_j'(\omega)+\beta_l'(\omega)}{2}\frac{\partial V_{j,l}^\varepsilon}{\partial t} =& \frac{ik^2}{2\varepsilon}\left(\frac{C_{j,j}(\frac{x}{\varepsilon^2})}{\beta_j}-\frac{C_{l,l}(\frac{x}{\varepsilon^2})}{\beta_l}\right)V_{j,l}^\varepsilon + \frac{ik^2}{2\varepsilon}\sum_{l'=0,l'\neq j}^{N-1}\frac{C_{j,l'}(\frac{x}{\varepsilon^2})}{\sqrt{\beta_{l'}\beta_j}}V_{l',l}^\varepsilon e^{i(\beta_{l'}-\beta_j)\frac{x}{\varepsilon^2}}\\ &-\frac{ik^2}{2\varepsilon}\sum_{l'=0,l'\neq l}^{N-1}\frac{C_{l,l'}(\frac{x}{\varepsilon^2})}{\sqrt{\beta_{l'}\beta_l}}V_{j,l'}^\varepsilon e^{i(\beta_l-\beta_{l'})\frac{x}{\varepsilon^2}} + \frac{ik^2}{2\varepsilon}\int_0^{n_1^2k^2}\frac{C_{j,\gamma'}(\frac{x}{\varepsilon^2})}{\sqrt[4]{\gamma'}\sqrt{\beta_j}}V_{\gamma',l}^\varepsilon e^{i(\sqrt{\gamma'}-\beta_j)\frac{x}{\varepsilon^2}}d\gamma'\\ &-\frac{ik^2}{2\varepsilon}\int_0^{n_1^2k^2}\frac{C_{l,\gamma'}(\frac{x}{\varepsilon^2})}{\sqrt[4]{\gamma'}\sqrt{\beta_l}}V_{j,\gamma'}^\varepsilon e^{-i(\sqrt{\gamma'}-\beta_l)\frac{x}{\varepsilon^2}}d\gamma'.\end{aligned}$$

The completion of the proof follows the reasoning in Ref. [11]. □

Using Proposition B.1, we then get the following expression for the Wigner transform (14):

$$W^{\rm s}(x,\kappa;t,\omega) = \sum_{j=0}^{N(\omega)-1} \frac{\partial_z\phi_j(0;\omega)^2}{\beta_j(\omega)} 2\pi W_j^{\rm c}(x;t,\omega)\delta(\kappa-\beta_j(\omega))$$
$$+ \sum_{j,l=0,j\neq l}^{N(\omega)-1} \frac{\partial_z\phi_j\partial_z\phi_l(0;\omega)}{\sqrt{\beta_j\beta_l(\omega)}} 2\pi W_{jl,{\rm s}}^{\rm c}\Big(t-\Big(\frac{1}{2v_j(\omega)}+\frac{1}{2v_l(\omega)}\Big)x,\omega\Big)$$
$$\times e^{-Q_{jl}(\omega)x}\delta\left(\kappa-\frac{\beta_j(\omega)+\beta_l(\omega)}{2}\right) e^{i(\beta_j(\omega)-\beta_l(\omega))\frac{x}{\varepsilon^2}}. \tag{105}$$

The second term decays exponentially with the propagation distance (with a decay rate that is related to the scattering mean free paths $1/\Lambda_j^s$ of the guided modes). If we neglect it, then we get the statement of Proposition 2.1.

B.2. Proof of Proposition 3.2

We consider the results of Proposition 2.1 and study the asymptotic regime $kD \gg 1$. We have

- $\beta_{j,D} \simeq \sqrt{n_1^2k^2-\pi^2(j-N)^2/D^2}$ and $\beta'_{j,D} \simeq n_1^2k/[c_o\sqrt{n_1^2k^2-\pi^2(j-N)^2/D^2}] = n_1^2k/(c_o\beta_{j,D})$, so that $1/v_{j_D,D} = \beta'_{j_D,D}$ converges to $n_1^2/(c_o\xi)$ when $\beta_{j_D,D}$ converges to $\sqrt{\gamma} = k\xi$.
- The expression (45) of $\tilde{\mathcal{R}}^\mu_{\xi\xi'}(x)$ comes from the explicit calculation of

$$\tilde{\mathcal{R}}^\mu_{\xi\xi'}(x) = D_0 \lim_{D\to+\infty}\frac{1}{D}\int_0^D\int_0^D \tilde\phi_\xi\tilde\phi_{\xi'}(z;\omega)\mathcal{R}^\mu(x,z-z')\tilde\phi_\xi\tilde\phi_{\xi'}(z';\omega)dzdz'.$$

- The function $\tilde\phi_\xi(z;\omega)$ has the following explicit form:

$$\tilde\phi_\xi(z;\omega) = \frac{\sqrt{r_{k^2\xi^2}(\omega)}}{\sqrt{\mathcal{N}(\xi^2)}}\psi_{k^2\xi^2}(z;\omega),$$

if $z\in(0,d)$, and

$$\tilde\phi_\xi(z;\omega) = \frac{\sqrt{r_{k^2\xi^2}(\omega)}}{\sqrt{\mathcal{N}(\xi^2)}}\Big\{\psi_{k^2\xi^2}(d;\omega)\cos\big[k\sqrt{n_1^2-\xi^2}(z-d)\big] + \frac{\partial_z\psi_{k^2\xi^2}(d;\omega)}{k\sqrt{n_1^2-\xi^2}}\sin\big[k\sqrt{n_1^2-\xi^2}(z-d)\big]\Big\},$$

if $z\in(d,+\infty)$, where ψ_γ is defined in (76) and

$$r_\gamma(\omega) = \frac{1}{\pi}\frac{\sqrt{n_1^2k^2-\gamma}}{(n_1^2k^2-\gamma)\psi_\gamma(d;\omega)^2+\partial_z\psi_\gamma(d;\omega)^2}.$$

For instance, if $\mathrm{n_b}(z)\equiv n_0$ for $z\in(0,d)$, then

$$\tilde\phi_\xi(z;\omega) = \sqrt{2k}\frac{\sqrt{n_1^2-\xi^2}\sin\big[kz\sqrt{n_0^2-\xi^2}\big]}{\sqrt{n_1^2-\xi^2+(n_0^2-n_1^2)\cos^2(kd\sqrt{n_0^2-\xi^2})}},$$

if $z\in(0,d)$, and

$$\tilde\phi_\xi(z;\omega) = \sqrt{2k}\frac{\sqrt{n_1^2-\xi^2}\sin\big[kd\sqrt{n_0^2-\xi^2}\big]\cos\big[k(z-d)\sqrt{n_1^2-\xi^2}\big]}{\sqrt{n_1^2-\xi^2+(n_0^2-n_1^2)\cos^2(kd\sqrt{n_0^2-\xi^2})}}$$
$$+\sqrt{2k}\frac{\sqrt{n_0^2-\xi^2}\cos\big[kd\sqrt{n_0^2-\xi^2}\big]\sin\big[k(z-d)\sqrt{n_1^2-\xi^2}\big]}{\sqrt{n_1^2-\xi^2+(n_0^2-n_1^2)\cos^2(kd\sqrt{n_0^2-\xi^2})}},$$

if $z\in(d,+\infty)$.

B.3. Proof of Proposition 4.1

We consider the spectral problem associated to the operator $\partial_z^2\phi + k^2 \mathrm{n}_{\mathrm{b},D}^2(z)\phi = \gamma\phi$ on $(0,D)$ with Dirichlet boundary condition at $z=0$ and Neumann boundary condition at $z=D$. Here $\mathrm{n}_{\mathrm{b},D}$ is defined by (64). We denote by $\gamma_{j,D}$ and $\phi_{j,D}$ the eigenvalues and orthonormal eigenfunctions as defined in Appendix A.1. Weyl's formula states that the number $M_D(\gamma)$ of eigenvalues larger than γ is

$$
\begin{aligned}
M_D(\gamma) &= \frac{D}{2\pi} \iint_{k^2(n_1^2 \mathbf{1}_{(0,\alpha)}(\zeta) + n_2^2 \mathbf{1}_{(\alpha,1)}(\zeta)) - \kappa^2 > \gamma} d\kappa d\zeta \big(1 + o(kD)\big) \\
&= kD \mathcal{M}\Big(\frac{\gamma}{k^2}\Big)\big(1+o(kD)\big),
\end{aligned}
$$

with

$$
\mathcal{M}(g) = \frac{\alpha}{\pi}\sqrt{n_1^2 - g}\mathbf{1}_{(n_2^2,n_1^2)}(g) + \frac{1-\alpha}{\pi}\sqrt{n_2^2 - g}\mathbf{1}_{(0,n_2^2)}(g),
$$

which gives (65)-(66), and also

$$
\beta_{j,D} \simeq k\mathcal{B}\Big(\frac{j-N}{kD}\Big),
$$

with

$$
\mathcal{B}^{-1}(\xi) = \mathcal{M}(\xi^2) = \frac{\alpha}{\pi}\sqrt{n_1^2 - \xi^2}\mathbf{1}_{(n_2,n_1)}(\xi) + \frac{1-\alpha}{\pi}\sqrt{n_2^2 - \xi^2}\mathbf{1}_{(0,n_2)}(\xi).
$$

We have

$$
\beta_{j,D}' \simeq \frac{1}{c_o}\big[\mathcal{B}(r) - r\partial_r \mathcal{B}(r)\big]_{r=\frac{j-N}{kD}},
$$

so that $1/v_{j_D,D}$ converges to $\frac{1}{c_o}[\mathcal{B}(r) - r\partial_r\mathcal{B}(r)]_{r=\mathcal{B}^{-1}(\xi)}$ when $\beta_{j_D,D}$ converges to $k\xi$, that is to say $1/v_{j_D,D}$ converges to

$$
\frac{1}{v_\xi} = \frac{1}{c_o}\Big(\xi - \frac{\mathcal{B}^{-1}(\xi)}{\partial_\xi \mathcal{B}^{-1}(\xi)}\Big),
$$

which gives (67).

The computation of the normalizing constants can be carried out as follows. For $\gamma \in (0, k^2 n_1^2)$ we denote by $\psi_{\gamma,D}$ the solution of $\partial_z^2 \psi_{\gamma,D} + k^2 \mathrm{n}_{\mathrm{b},D}^2(z)\psi_{\gamma,D} = \gamma\psi_{\gamma,D}$ starting from $\psi_{\gamma,D}(0) = 0$ and $\partial_z\psi_{\gamma,D}(0) = 1$. We have $\phi_{j,D} = \sqrt{r_{j,D}}\psi_{\gamma_{j,D},D}$ with $r_{j,D}^{-1} = \int_0^D \psi_{\gamma_{j,D},D}(z)^2 dz$.

If $\gamma \in (k^2 n_2^2, k^2 n_1^2)$, then we have $\psi_{\gamma,D}(z) = \psi_\gamma(z)$ for any $z \le d$, where ψ_γ has been defined by (76),

$$
\psi_{\gamma,D}(z) = \psi_\gamma(d)\cos\big(\sqrt{n_1^2k^2 - \gamma}(z-d)\big) + \frac{\partial_z\psi_\gamma(d)}{\sqrt{n_1^2k^2-\gamma}}\sin\big(\sqrt{n_1^2k^2-\gamma}(z-d)\big)
$$

if $z \in (d, d+\alpha(D-d))$, and

$$
\begin{aligned}
\psi_{\gamma,D}(z) =& \psi_{\gamma,D}(d+\alpha(D-d))\cosh\big(\sqrt{\gamma - n_2^2k^2}(z-d-\alpha(D-d))\big) \\
&+ \frac{\partial_z\psi_{\gamma,D}(d+\alpha(D-d))}{\sqrt{\gamma-n_2^2k^2}}\sinh\big(\sqrt{\gamma-n_2^2k^2}(z-d-\alpha(D-d))\big)
\end{aligned}
$$

if $z \in (d+\alpha(D-d), D)$. We then obtain that, if $\gamma_{j_D,D}$ converges to γ, then $r_{j_D,D}$ has the asymptotic form

$$
\frac{1}{Dr_{j_D,D}} \stackrel{D\to+\infty}{\longrightarrow} \frac{\alpha}{2}\Big[\psi_\gamma(d)^2 + \frac{\partial_z\psi_\gamma(d)^2}{n_1^2k^2-\gamma}\Big].
$$

If $\gamma \in (0, k^2 n_2^2)$, then we have

$$
\psi_{\gamma,D}(z) = \psi_{\gamma,D}(d)\cos\big(\sqrt{n_1^2k^2-\gamma}(z-d)\big) + \frac{\partial_z\psi_{\gamma,D}(d)}{\sqrt{n_1^2k^2-\gamma}}\sin\big(\sqrt{n_1^2k^2-\gamma}(z-d)\big)
$$

if $z \in (d, d+\alpha(D-d))$ and

$$\psi_{\gamma,D}(z) = \psi_{\gamma,D}(d+\alpha(D-d)) \cos\big(\sqrt{n_2^2k^2-\gamma}(z-d-\alpha(D-d))\big) + \frac{\partial_z\psi_{\gamma,D}(d+\alpha(D-d))}{\sqrt{n_2^2k^2-\gamma}} \sin\big(\sqrt{n_2^2k^2-\gamma}(z-d-\alpha(D-d))\big)$$

if $z \in (d+\alpha(D-d), D)$. We find that, if $\gamma_{j_D,D}$ converges to γ, then $r_{j_D,D}$ has the asymptotic form

$$\frac{1}{Dr_{j_D,D}} \stackrel{D\to+\infty}{\longrightarrow} \Big[\frac{\alpha}{2} + \frac{1-\alpha}{4}\frac{(n_1^2+n_2^2)k^2-2\gamma}{n_2^2k^2-\gamma}\Big]\Big[\psi_\gamma(d)^2 + \frac{\partial_z\psi_\gamma(d)^2}{n_1^2k^2-\gamma}\Big].$$

The expression (68) for $\tilde{\mathcal{R}}^\mu_{\xi\xi'}(x)$ is obtained from the explicit calculation of

$$\lim_{D\to+\infty}\frac{1}{D}\int_0^D\int_0^D \phi_{j_D,D}\phi_{j'_D,D}(z;\omega)\mathcal{R}^\mu(x,z-z')\phi_{j_D,D}\phi_{j'_D,D}(z';\omega)dzdz',$$

when $\gamma_{j_D,D}\to k^2\xi^2$ and $\gamma_{j'_D,D}\to k^2\xi'^2$.

B.4. Proof of Proposition 4.2

Let us consider the spectral problem associated to the operator $\partial_z^2\phi + k^2\mathrm{n}_{\mathrm{b},D}^2(z)\phi = \gamma\phi$ on $(0,D)$ with Dirichlet boundary condition at $z=0$ and Neumann boundary condition at $z=D$. Here, $\mathrm{n}_{\mathrm{b},D}$ is defined by (63) and $\tilde{\mathrm{n}}_\mathrm{b}$ is smooth and nonincreasing. We denote by $\gamma_{j,D}$ and $\phi_{j,D}$ the eigenvalues and orthonormal eigenfunctions as defined in Appendix A.1. Weyl's formula states that the number $M_D(\gamma)$ of eigenvalues larger than γ is

$$\begin{aligned} M_D(\gamma) &= \frac{D}{2\pi}\iint_{k^2\tilde{\mathrm{n}}_\mathrm{b}^2(\zeta)-\kappa^2>\gamma} d\kappa d\zeta\big(1+o(kD)\big) \\ &= kD\mathcal{M}\Big(\frac{\gamma}{k^2}\Big)\big(1+o(kD)\big), \end{aligned}$$

with $\mathcal{M}(g)$ defined by (73). This gives the expression (70) of the density of states. We also have $\beta_{j,D}\simeq k\mathcal{B}\big(\frac{j-N}{kD}\big)$, with $\mathcal{B}^{-1}(\xi)=\mathcal{M}(\xi^2)$. Consequently $\beta'_{j,D}\simeq(1/c_o)[\mathcal{B}(r)-r\partial_r\mathcal{B}(r)]_{r=\frac{j-N}{kD}}$, so that $1/v_{j_D,D}$ converges to $(1/c_o)[\mathcal{B}(r)-r\partial_r\mathcal{B}(r)]_{r=\mathcal{B}^{-1}(\xi)}$ when $\beta_{j_D,D}$ converges to $k\xi$, that is to say $1/v_{j,D}$ converges to

$$\frac{1}{v_\xi} = \frac{1}{c_o}\Big(\xi - \frac{\mathcal{B}^{-1}(\xi)}{\partial_\xi\mathcal{B}^{-1}(\xi)}\Big),$$

which gives (72).

The normalizing constants can be computed as follows. For $\gamma\in(0,k^2n_1^2)$ we denote by $\psi_{\gamma,D}$ the solution of $\partial_z^2\psi_{\gamma,D}+k^2\mathrm{n}_{\mathrm{b},D}^2(z)\psi_{\gamma,D}=\gamma\psi_{\gamma,D}$ starting from $\psi_{\gamma,D}(0)=0$ and $\partial_z\psi_{\gamma,D}(0)=1$. We have $\phi_{j,D}=\sqrt{r_{j,D}}\psi_{\gamma_{j,D},D}$ with $r_{j,D}^{-1}=\int_0^D\psi_{\gamma_{j,D},D}(z)^2dz$. The goal is to study the asymptotic form of the normalizing constant $r_{j,D}$ as $D\to+\infty$.

The idea is to discretize the smooth function $\tilde{\mathrm{n}}_\mathrm{b}(\zeta)$. We fix an integer M and consider the following discretized version of $\tilde{\mathrm{n}}_\mathrm{b}$:

$$\tilde{n}_\mathrm{b}^M(\zeta) = \sum_{j=0}^{M-1}\tilde{\mathrm{n}}_\mathrm{b}\Big(\frac{j}{M}\Big)\mathbf{1}_{[j/M,(j+1)/M)}(\zeta),$$

and the associated $n_{\mathrm{b},D}^M$ defined as in (62) in terms of $\tilde{n}_\mathrm{b}^M$. We study the function $\psi_{\gamma,D}^M$ defined as the solution of $\partial_z^2\psi_{\gamma,D}^M+k^2\mathrm{n}_{\mathrm{b},D}^M(z)^2\psi_{\gamma,D}^M=\gamma\psi_{\gamma,D}^M$ starting from $\psi_{\gamma,D}^M(0)=0$ and $\partial_z\psi_{\gamma,D}^M(0)=1$. We introduce $z_j^M=d+(j/M)(D-d)$ for $j=0,\ldots,M$. If $\gamma\in(k^2\tilde{\mathrm{n}}_\mathrm{b}((j+1)/M),k^2\tilde{\mathrm{n}}_\mathrm{b}(j/M))$ for some j, then we have $\psi_{\gamma,D}^M(z)=\psi_\gamma(z)$ for any $z\le z_0^M=d$, where ψ_γ has been defined by (76), we have

$$\psi_{\gamma,D}^M(z) = \psi_\gamma(d)\cos\big(\sqrt{\tilde{\mathrm{n}}_\mathrm{b}^2(0)k^2-\gamma}(z-d)\big) + \frac{\partial_z\psi_\gamma(d)}{\sqrt{\tilde{\mathrm{n}}_\mathrm{b}^2(0)k^2-\gamma}}\sin\big(\sqrt{\tilde{\mathrm{n}}_\mathrm{b}^2(0)k^2-\gamma}(z-d)\big)$$

if $z \in (z_0^M, z_1^M)$, and we have

$$\psi_{\gamma,D}^M(z) = \psi_{\gamma,D}^M(z_{l-1}^M)\cos\Big(\sqrt{\tilde{n}_b^2((l-1)/M)k^2-\gamma}(z-z_{l-1}^M)\Big) + \frac{\partial_z\psi_{\gamma,D}^M(z_{l-1}^m)}{\sqrt{\tilde{n}_b^2((l-1)/M)k^2-\gamma}}\sin\Big(\sqrt{\tilde{n}_b^2((l-1)/M)k^2-\gamma}(z-z_{l-1}^M)\Big)$$

if $z \in (z_{l-1}^M, z_l^M)$ for $l \leq j$. The function $\psi_{\gamma,D}^M$ decays exponentially in z for $z > z_j^M$. We then obtain that

$$\frac{1}{D}\int_0^D \psi_{\gamma,D}^M(z)^2 dz \stackrel{D\to+\infty}{\longrightarrow} \frac{1}{2M}\Big[1+\sum_{l=1}^{j-1}\prod_{l'=1}^{l}\Big(1+\frac{1}{2}\frac{k^2\big(\tilde{n}_b^2((l'-1)/M)-\tilde{n}_b^2(l'/M)\big)}{k^2\tilde{n}_b^2(l'/M)-\gamma}\Big)\Big]\Big[\psi_\gamma(d)^2+\frac{\partial_z\psi_\gamma(d)^2}{n_1^2k^2-\gamma}\Big].$$

If $\tilde{n}_b$ is smooth and if $\gamma \in (0, k^2n_2^2)$, then we get for large M

$$\lim_{D\to+\infty}\frac{1}{D}\int_0^D \psi_{\gamma,D}^M(z)^2 dz \simeq \frac{1}{2M}\Big[1+\sum_{l=1}^{M-1}\exp\Big(\frac{1}{2}\int_{\tilde{n}_b^2(l/M)}^{\tilde{n}_b^2(0)}\frac{k^2}{k^2s-\gamma}ds\Big)\Big]\Big[\psi_\gamma(d)^2+\frac{\partial_z\psi_\gamma(d)^2}{n_1^2k^2-\gamma}\Big].$$

If $\gamma \in (k^2n_2^2, k^2n_1^2)$, then

$$\lim_{D\to+\infty}\frac{1}{D}\int_0^D \psi_{\gamma,D}^M(z)^2 dz \simeq \frac{1}{2M}\Big[1+\sum_{l=1}^{j(\gamma)-1}\exp\Big(\frac{1}{2}\int_{\tilde{n}_b^2(l/M)}^{\tilde{n}_b^2(0)}\frac{k^2}{k^2s-\gamma}ds\Big)\Big]\Big[\psi_\gamma(d)^2+\frac{\partial_z\psi_\gamma(d)^2}{n_1^2k^2-\gamma}\Big].$$

where $j(\gamma)$ is the unique j such that $\gamma \in (k^2\tilde{n}_b^2((j+1)/M), k^2\tilde{n}_b^2(j/M))$. This gives

$$\lim_{M,D\to+\infty}\frac{1}{D}\int_0^D \psi_{\gamma,D}^M(z)^2 dz \simeq \frac{1}{2}\Big[\int_0^{\tilde{n}_b^{-1}(\frac{\sqrt{\gamma}}{k})}\frac{\sqrt{k^2\tilde{n}_b^2(0)-\gamma}}{\sqrt{k^2\tilde{n}_b^2(\zeta)-\gamma}}d\zeta\Big]\Big[\psi_\gamma(d)^2+\frac{\partial_z\psi_\gamma(d)^2}{n_1^2k^2-\gamma}\Big],$$

where $\zeta = \tilde{n}_b^{-1}(\xi)$ is the smallest $\zeta \in (0,1)$ such that $\tilde{n}_b(\zeta) = \xi$. Finally we get that $r_{j_D,D}$ has the asymptotic form

$$\frac{1}{Dr_{j_D,D}} \stackrel{D\to+\infty}{\longrightarrow} \frac{1}{2}\Big[\int_0^{\tilde{n}_b^{-1}(\xi)}\frac{1}{\sqrt{\tilde{n}_b^2(\zeta)-\xi^2}}d\zeta\Big]\Big[\sqrt{n_1^2-\xi^2}\psi_\gamma(d)^2+\frac{\partial_z\psi_\gamma(d)^2}{k^2\sqrt{n_1^2-\xi^2}}\Big],$$

when $\gamma_{j_D,D} \to k^2\xi^2$. The expression (74) of $\tilde{\mathcal{R}}_{\xi\xi'}^\mu(x)$ then comes from the explicit calculation of

$$\lim_{D\to+\infty}\frac{1}{D}\int_0^D\int_0^D \phi_{j_D,D}\phi_{j'_D,D}(z;\omega)\mathcal{R}^\mu(x,z-z')\phi_{j_D,D}\phi_{j'_D,D}(z';\omega)dzdz',$$

when $\gamma_{j_D,D} \to k^2\xi^2$ and $\gamma_{j'_D,D} \to k^2\xi'^2$.

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Maarten V. de Hoop
Simons Chair in Computational and Applied Mathematics and Earth Science, Rice University, Houston, TX 77005
e-mail: `mdehoop@rice.edu`

Josselin Garnier
CMAP, CNRS, Ecole polytechnique, Institut Polytechnique de Paris, 91128 Palaiseau Cedex, France
e-mail: `josselin.garnier@polytechnique.edu`

Knut Sølna
Department of Mathematics, University of California, Irvine CA 92697
e-mail: `ksolna@math.uci.edu`